

Duals of coloured quantum universal enveloping algebras and coloured universal \mathcal{T} -matrices

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Abstract

We extend the notion of dually conjugate Hopf (super)algebras to the coloured Hopf (super)algebras \mathcal{H}^c that we recently introduced. We show that if the standard Hopf (super)algebras \mathcal{H}_q that are the building blocks of \mathcal{H}^c have Hopf duals \mathcal{H}_q^* , then the latter may be used to construct coloured Hopf duals \mathcal{H}^{c*} , endowed with coloured algebra and antipode maps, but with a standard coalgebraic structure. Next, we review the case where the \mathcal{H}_q 's are quantum universal enveloping algebras of Lie (super)algebras $U_q(g)$, so that the corresponding \mathcal{H}_q^* 's are quantum (super)groups G_q . We extend the Fronsdal and Galindo universal \mathcal{T} -matrix formalism to the coloured pairs $(U^c(g), G^c)$ by defining coloured universal \mathcal{T} -matrices. We then show that together with the coloured universal \mathcal{R} -matrices previously introduced, the latter provide an algebraic formulation of the coloured RTT -relations, proposed by Basu-Mallick. This establishes a link between the coloured extensions of Drinfeld-Jimbo and Faddeev-Reshetikhin-Takhtajan pictures of quantum groups and quantum algebras. Finally, we illustrate the construction of coloured pairs by giving some explicit results for the two-parameter deformations of $(U(gl(2)), Gl(2))$, and $(U(gl(1/1)), Gl(1/1))$.

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I INTRODUCTION

In a recent paper [1] (henceforth referred to as I and whose equations will be subsequently quoted by their number preceded by I), we did introduce some new algebraic structures $\mathcal{H}^c = (\mathcal{H}, \mathcal{C}, \mathcal{G})$, termed coloured Hopf algebras (see also Ref. [2]). They are constructed by starting from a standard Hopf algebra set $\mathcal{H} = \{ \mathcal{H}_q \mid q \in \mathcal{Q} \}$, corresponding to some parameter set \mathcal{Q} , and by generalizing the definitions of coalgebra maps, and antipodes, by combining the standard ones with the transformations of an algebra isomorphism group \mathcal{G} , called colour group. Such transformations are labelled by some colour parameters, taking values in a colour set \mathcal{C} . Whenever the starting Hopf algebras \mathcal{H}_q are quasitriangular, the resulting coloured one \mathcal{H}^c is characterized by the existence of a coloured universal \mathcal{R} -matrix, denoted by \mathcal{R}^c , and satisfying the coloured Yang-Baxter equation (YBE), i.e., the YBE with nonadditive spectral parameters [3, 4].

In I, we applied these new concepts to the Drinfeld-Jimbo formulation of quantum algebras [5], and constructed various examples of coloured quantum universal enveloping algebras (QUEA's) of both semisimple and nonsemisimple Lie algebras.

In a more recent paper [6] (henceforth referred to as II), we showed that our definitions can be easily extended to deal with graded algebraic structures [7], thereby leading to coloured Hopf superalgebras in general, and to coloured QUEA's of Lie superalgebras in particular. In such cases, quasitriangularity implies that \mathcal{R}^c is a solution of the coloured graded YBE.

In the literature, there exist other extensions of Hopf algebras related to the coloured YBE, and using either the Drinfeld-Jimbo [5], or the Faddeev-Reshetikhin-Takhtajan [8] approach to quantum groups and quantum algebras. In the former class, one should mention two previous introductions of some algebraic structures called coloured Hopf algebras, the first one by Ohtsuki [9] in the context of knot theory, and the second one by Bonatsos *et al* [10] in the study of some nonlinear deformation of $su(2)$. In the latter class, some extensions of the Faddeev-Reshetikhin-Takhtajan RLL - and RTT -relations, where R is replaced by a coloured R -matrix, have been considered by Kundu and Basu-Mallick [11, 12], thereby leading to coloured $U(R)$ and $A(R)$ Hopf algebras, respectively.

As shown in I, some direct connections exist between our coloured Hopf algebras, and those previously introduced by Ohtsuki [9], and by Bonatsos *et al* [10], as the former may

be considered as generalizations of the latter. On the contrary, possible relationships with coloured $U(R)$ and $A(R)$ Hopf algebras have not been investigated so far.

It is one of the aims of the present paper to fill in this gap for the coloured $A(R)$ algebras, introduced by Basu-Mallick [12]. For such a purpose, we shall have to extend the notion of dually conjugate Hopf algebras [13] to the coloured context.

In the case of standard Hopf algebras, the interest of such a duality concept was recently highlighted by Fronsdal and Galindo [14], who constructed dual bases for the two-parameter deformations of $Gl(2)$ and $U(gl(2))$, and used them to build a universal \mathcal{T} -matrix, providing an algebraic formulation of the RTT -relations, and yielding the quantum group generalization of the familiar exponential map from a Lie algebra to a Lie group. A similar construction was carried out for some other quantum group and algebra pairs [15, 16, 17, 18], and used to get representations of one member of the pair from those of the other [19, 20]. Moreover, the Fronsdal and Galindo approach was also recently generalized to quantum supergroups and superalgebras [21].

We plan to show here that this universal \mathcal{T} -matrix formalism can be extended to coloured QUEA's and their duals, and that the resulting coloured universal \mathcal{T} -matrix, to be denoted by \mathcal{T}^c , provides the key notion for establishing a link between such duals and Basu-Mallick's coloured $A(R)$ algebras.

This paper is organized as follows. In Sec. II, duals \mathcal{H}^{c*} of coloured Hopf (super)algebras \mathcal{H}^c are defined. In Sec. III, the case of coloured QUEA's of Lie (super)algebras is considered, and coloured universal \mathcal{T} -matrices are introduced. In Secs. IV and V, the theory developed in the previous Section is applied to construct pairs $(\mathcal{H}^c, \mathcal{H}^{c*})$ for the two-parameter deformations of $(U(gl(2)), Gl(2))$, and $(U(gl(1/1)), Gl(1/1))$, respectively. Section VI contains some concluding remarks.

II DUALS OF COLOURED HOPF (SUPER)ALGEBRAS

Let $\mathcal{H}^c = (\mathcal{H}, \mathcal{C}, \mathcal{G}) = (\mathcal{H}_q, +, m_q, \iota_q, \Delta_{q,\nu}^{\lambda,\mu}, \epsilon_{q,\nu}, S_{q,\nu}^\mu; k, \mathcal{Q}, \mathcal{C}, \mathcal{G})$ be a coloured Hopf algebra over some field k ($= \mathbb{C}$ or \mathbb{R}) [1, 2]. Here \mathcal{Q} , \mathcal{C} , and $\mathcal{G} = \{ \sigma^\nu : \mathcal{H}_q \rightarrow \mathcal{H}_{q^\nu} \mid q, q^\nu \in \mathcal{Q}, \nu \in \mathcal{C} \}$ denote the parameter set, the colour set, and the colour group, respectively. Hence, by definition, the σ^ν 's satisfy Eqs. (I2.1)–(I2.4). The maps $m_q : \mathcal{H}_q \otimes \mathcal{H}_q \rightarrow \mathcal{H}_q$, and $\iota_q : k \rightarrow \mathcal{H}_q$

are standard multiplication and unit maps, whereas $\Delta_{q,\nu}^{\lambda,\mu} \equiv (\sigma^\lambda \otimes \sigma^\mu) \circ \Delta_q \circ \sigma_\nu : \mathcal{H}_{q^\nu} \rightarrow \mathcal{H}_{q^\lambda} \otimes \mathcal{H}_{q^\mu}$, $\epsilon_{q,\nu} \equiv \epsilon_q \circ \sigma_\nu : \mathcal{H}_{q^\nu} \rightarrow k$, and $S_{q,\nu}^\mu \equiv \sigma^\mu \circ S_q \circ \sigma_\nu : \mathcal{H}_{q^\nu} \rightarrow \mathcal{H}_{q^\mu}$, defined in terms of standard comultiplication Δ_q , counit ϵ_q , and antipode S_q , with $\sigma_\nu \equiv (\sigma^\nu)^{-1}$, are called coloured comultiplication, counit, and antipode maps, respectively, and satisfy some generalized axioms, stated in Proposition II.2 of I.

Let us now assume that the standard Hopf algebras \mathcal{H}_q , belonging to \mathcal{H} , have Hopf duals $(\mathcal{H}_q^*, +, \tilde{m}_q, \tilde{\iota}_q, \tilde{\Delta}_q, \tilde{\epsilon}_q, \tilde{S}_q; k)$ (or in short \mathcal{H}_q^*), and let us set $\mathcal{H}^* = \{\mathcal{H}_q^* \mid q \in \mathcal{Q}\}$. This means [13] that for any $q \in \mathcal{Q}$, there exists a doubly nondegenerate bilinear form $\langle \cdot, \cdot \rangle_q$, such that the bialgebra and antipode maps of the dual pair $(\mathcal{H}_q, \mathcal{H}_q^*)$ are related by

$$\langle \tilde{m}_q(x \otimes y), X \rangle_q = \langle x \otimes y, \Delta_q(X) \rangle_q, \quad \langle \tilde{\iota}_q(1), X \rangle_q = \langle \tilde{1}_q, X \rangle_q = \epsilon_q(X), \quad (2.1)$$

$$\langle \tilde{\Delta}_q(x), X \otimes Y \rangle_q = \langle x, m_q(X \otimes Y) \rangle_q, \quad \tilde{\epsilon}_q(x) = \langle x, \iota_q(1) \rangle_q = \langle x, 1_q \rangle_q, \quad (2.2)$$

$$\langle \tilde{S}_q(x), X \rangle_q = \langle x, S_q(X) \rangle_q, \quad (2.3)$$

for any $x, y \in \mathcal{H}_q^*$, and any $X, Y \in \mathcal{H}_q$.

Let us next introduce

Definition II.1 Let $\rho^\nu : \mathcal{H}_q^* \rightarrow \mathcal{H}_{q^\nu}^*$ be defined by

$$\langle \rho^\nu(x), \sigma^\nu(X) \rangle_{q^\nu} = \langle x, X \rangle_q, \quad (2.4)$$

for any $x \in \mathcal{H}_q^*$, $X \in \mathcal{H}_q$, $q \in \mathcal{Q}$, and $\nu \in \mathcal{C}$.

From the properties of the σ^ν 's and the nondegeneracy of $\langle \cdot, \cdot \rangle_q$, it is then straightforward to show

Proposition II.2 The maps ρ^ν , defined in Eq. (2.4), are coalgebra isomorphisms, and the set $\{\rho^\nu \mid \nu \in \mathcal{C}\}$ is a group isomorphic to \mathcal{G} (and denoted by the same symbol) with respect to the composition of maps. In other words, the ρ^ν 's are one-to-one, and satisfy the properties

$$\tilde{\Delta}_{q^\nu} \circ \rho^\nu = (\rho^\nu \otimes \rho^\nu) \circ \tilde{\Delta}_q, \quad \tilde{\epsilon}_{q^\nu} \circ \rho^\nu = \tilde{\epsilon}_q, \quad (2.5)$$

and

$$\begin{aligned} \forall \nu, \nu' \in \mathcal{C} : \rho^{\nu' \circ \nu} &= \rho^{\nu'} \circ \rho^\nu : \mathcal{H}_q^* \rightarrow \mathcal{H}_{q^{\nu' \circ \nu}}^*, \\ \rho^{\nu^0} &= \text{id} : \mathcal{H}_q^* \rightarrow \mathcal{H}_{q^{\nu^0}}^* = \mathcal{H}_q^*, \\ \forall \nu \in \mathcal{C} : \rho^{\nu^i} &= \rho_\nu \equiv (\rho^\nu)^{-1} : \mathcal{H}_{q^\nu}^* \rightarrow \mathcal{H}_q^*, \end{aligned} \quad (2.6)$$

where $\nu' \circ \nu$, ν^0 , and ν^i have the same meaning as in Eqs. (I2.2)–(I2.4).

Proceeding as for \mathcal{H} in I, we may now combine the definitions of \mathcal{H}^* , \mathcal{C} , and \mathcal{G} into

Definition II.3 *The maps $\tilde{m}_{q,\lambda,\mu}^\nu : \mathcal{H}_{q^\lambda}^* \otimes \mathcal{H}_{q^\mu}^* \rightarrow \mathcal{H}_{q^\nu}^*$, $\tilde{l}_q^\nu : k \rightarrow \mathcal{H}_{q^\nu}^*$, and $\tilde{S}_{q,\mu}^\nu : \mathcal{H}_{q^\mu}^* \rightarrow \mathcal{H}_{q^\nu}^*$, defined by*

$$\tilde{m}_{q,\lambda,\mu}^\nu \equiv \rho^\nu \circ \tilde{m}_q \circ (\rho_\lambda \otimes \rho_\mu), \quad \tilde{l}_q^\nu \equiv \rho^\nu \circ \tilde{l}_q, \quad \tilde{S}_{q,\mu}^\nu \equiv \rho^\nu \circ \tilde{S}_q \circ \rho_\mu, \quad (2.7)$$

for any $q \in \mathcal{Q}$, and any $\lambda, \mu, \nu \in \mathcal{C}$, are called coloured multiplication, unit, and antipode, respectively.

By using the dual pairing (2.1)–(2.3), Definitions II.1 and II.3, as well as Definition II.1 and Proposition II.2 of I, it is easy to establish

Proposition II.4 *The coloured multiplication, unit, and antipode maps, defined in Eq. (2.7), are dual to the coloured comultiplication, counit, and antipode of \mathcal{H}^c , i.e.,*

$$\begin{aligned} \forall x \in \mathcal{H}_{q^\lambda}^*, \forall y \in \mathcal{H}_{q^\mu}^*, \forall X \in \mathcal{H}_{q^\nu}^* : \langle \tilde{m}_{q,\lambda,\mu}^\nu(x \otimes y), X \rangle_{q^\nu} &= \langle x \otimes y, \Delta_{q,\nu}^{\lambda,\mu}(X) \rangle_{q^\lambda, q^\mu}, \\ \forall X \in \mathcal{H}_{q^\nu}^* : \langle \tilde{l}_q^\nu(1), X \rangle_{q^\nu} &= \epsilon_{q,\nu}(X), \\ \forall x \in \mathcal{H}_{q^\mu}^*, \forall X \in \mathcal{H}_{q^\nu}^* : \langle \tilde{S}_{q,\mu}^\nu(x), X \rangle_{q^\nu} &= \langle x, S_{q,\nu}^\mu(X) \rangle_{q^\mu}. \end{aligned} \quad (2.8)$$

In addition, they transform under \mathcal{G} as

$$\begin{aligned} \tilde{m}_{q,\alpha,\beta}^\nu \circ (\rho_\alpha^\lambda \otimes \rho_\beta^\mu) &= \tilde{m}_{q,\lambda,\mu}^\nu = \rho_\gamma^\nu \circ \tilde{m}_{q,\lambda,\mu}^\gamma, \\ \rho_\alpha^\nu \circ \tilde{l}_q^\alpha &= \tilde{l}_q^\nu, \\ \tilde{S}_{q,\alpha}^\nu \circ \rho_\mu^\alpha &= \tilde{S}_{q,\mu}^\nu = \rho_\beta^\nu \circ \tilde{S}_{q,\mu}^\beta, \end{aligned} \quad (2.9)$$

and satisfy generalized associativity, unit, and antipode axioms

$$\begin{aligned} \tilde{m}_{q,\lambda,\mu}^\nu \circ (\tilde{m}_{q,\alpha,\beta}^\lambda \otimes \rho_\gamma^\mu) &= \tilde{m}_{q,\lambda',\mu'}^\nu \circ (\rho_\alpha^{\lambda'} \otimes \tilde{m}_{q,\beta,\gamma}^{\mu'}), \\ \tilde{m}_{q,\lambda,\mu}^\nu \circ (\tilde{l}_q^\lambda \otimes \rho_\alpha^\mu) &= \tilde{m}_{q,\lambda',\mu'}^\nu \circ (\rho_\alpha^{\lambda'} \otimes \tilde{l}_q^{\mu'}) = \rho_\alpha^\nu, \\ \tilde{m}_{q,\lambda,\mu}^\nu \circ (\tilde{S}_{q,\alpha}^\lambda \otimes \rho_\alpha^\mu) \circ \tilde{\Delta}_{q^\alpha} &= \tilde{m}_{q,\lambda',\mu'}^\nu \circ (\rho_\alpha^{\lambda'} \otimes \tilde{S}_{q,\alpha}^{\mu'}) \circ \tilde{\Delta}_{q^\alpha} = \tilde{l}_q^\nu \circ \tilde{\epsilon}_{q^\alpha}, \end{aligned} \quad (2.10)$$

as well as generalized bialgebra axioms

$$\begin{aligned} \tilde{\Delta}_{q^\nu} \circ \tilde{m}_{q,\lambda,\mu}^\nu &= (\tilde{m}_{q,\lambda,\mu}^\nu \otimes \tilde{m}_{q,\lambda,\mu}^\nu) \circ (\text{id} \otimes \tau \otimes \text{id}) \circ (\tilde{\Delta}_{q^\lambda} \otimes \tilde{\Delta}_{q^\mu}), \\ \tilde{\Delta}_{q^\nu} \circ \tilde{l}_q^\nu &= \tilde{l}_q^\nu \otimes \tilde{l}_q^\nu, \\ \tilde{\epsilon}_{q^\nu} \circ \tilde{m}_{q,\lambda,\mu}^\nu &= \tilde{\epsilon}_{q^\lambda} \otimes \tilde{\epsilon}_{q^\mu}, \\ \tilde{\epsilon}_{q^\nu} \circ \tilde{l}_q^\nu &= \text{id}, \end{aligned} \quad (2.11)$$

where $\rho_\mu^\lambda \equiv \rho^\lambda \circ \rho_\mu$, τ is the twist map, and no summation is implied over repeated indices.

Remarks. (1) As usual for Hopf algebras, the twist map is defined by $\tau(x \otimes y) = y \otimes x$ for any x, y belonging to appropriate spaces. (2) As shown in Eqs. (2.10) and (I2.7), the coloured antipodes $\tilde{S}_{q,\mu}^\nu$ and $S_{q,\nu}^\mu$ satisfy distinct generalized axioms. A similar remark is valid for the generalized bialgebra structures given in Eqs. (2.11) and (I2.8), respectively.

From Proposition II.4, one obtains

Corollary II.5 *For any $q \in \mathcal{Q}$, any $\nu \in \mathcal{C}$, and $q_\nu \equiv q^{\nu^i}$, $(\mathcal{H}_q^*, +, \tilde{m}_{q_\nu, \nu, \nu}^\nu, \tilde{\iota}_{q_\nu}^\nu, \tilde{\Delta}_q, \tilde{\epsilon}_q, \tilde{S}_{q_\nu, \nu}^\nu; k)$ is a Hopf algebra over k with multiplication $\tilde{m}_{q_\nu, \nu, \nu}^\nu$, unit $\tilde{\iota}_{q_\nu}^\nu$, and antipode $\tilde{S}_{q_\nu, \nu}^\nu$, defined by particularizing Eq. (2.7). Moreover, it is the Hopf dual of the Hopf algebra $(\mathcal{H}_q, +, m_q, \iota_q, \Delta_{q_\nu, \nu}^{\nu, \nu}, \epsilon_{q_\nu, \nu}, S_{q_\nu, \nu}^\nu; k)$, considered in Corollary II.3 of I.*

Remark. In particular, for $\nu = \nu^0$, we get back the original Hopf structures of \mathcal{H}_q^* and \mathcal{H}_q .

This result can be generalized as follows:

Definition II.6 *If the standard Hopf algebras \mathcal{H}_q that are the building blocks of a coloured Hopf algebra \mathcal{H}^c have Hopf duals \mathcal{H}_q^* , then the set $\mathcal{H}^* = \{\mathcal{H}_q^* \mid q \in \mathcal{Q}\}$, endowed with coloured multiplication, unit, and antipode maps $\tilde{m}_{q, \lambda, \mu}^\nu$, $\tilde{\iota}_q^\nu$, $\tilde{S}_{q, \mu}^\nu$, as defined in (2.7), is called coloured Hopf dual of \mathcal{H}^c , and denoted by any one of the symbols $(\mathcal{H}_q^*, +, \tilde{m}_{q, \lambda, \mu}^\nu, \tilde{\iota}_q^\nu, \tilde{\Delta}_q, \tilde{\epsilon}_q, \tilde{S}_{q, \mu}^\nu; k, \mathcal{Q}, \mathcal{C}, \mathcal{G})$, $(\mathcal{H}^*, \mathcal{C}, \mathcal{G})$, or \mathcal{H}^{c*} .*

The coloured antipode $\tilde{S}_{q, \mu}^\nu$ satisfies some additional properties, which are again distinct from the corresponding ones of $S_{q, \nu}^\mu$, given in Proposition II.5 of I.

Proposition II.7 *The coloured antipode $\tilde{S}_{q, \mu}^\nu$ of a coloured Hopf dual \mathcal{H}^{c*} fulfils the relations*

$$\begin{aligned} \tilde{S}_{q, \gamma}^\nu \circ \tilde{m}_{q, \alpha, \beta}^\gamma &= \tilde{m}_{q, \mu, \lambda}^\nu \circ \tau \circ (\tilde{S}_{q, \alpha}^\lambda \otimes \tilde{S}_{q, \beta}^\mu), & \tilde{S}_{q, \mu}^\nu \circ \tilde{\iota}_q^\mu &= \tilde{\iota}_q^\nu, \\ (\tilde{S}_{q, \mu}^\nu \otimes \tilde{S}_{q, \mu}^\nu) \circ \tilde{\Delta}_{q^\mu} &= \tau \circ \tilde{\Delta}_{q^\nu} \circ \tilde{S}_{q, \mu}^\nu, & \tilde{\epsilon}_{q^\nu} \circ \tilde{S}_{q, \mu}^\nu &= \tilde{\epsilon}_{q^\mu}. \end{aligned} \quad (2.12)$$

The case where the \mathcal{H}_q 's are Hopf superalgebras $(\mathcal{H}_q, +, \gamma_q, m_q, \iota_q, \Delta_q, \epsilon_q, S_q; k)$ [7] can be dealt with only some minor changes. Here $\gamma_q : \mathcal{H}_q \rightarrow \mathcal{H}_q$ denotes their grading automorphism, i.e.,

$$\gamma_q(X) = (-1)^{\deg X} X, \quad (2.13)$$

for any homogeneous $X \in \mathcal{H}_q$, where $\deg X = 0$ or 1 according to whether X is even or odd. As all vector spaces are now graded, the tensor product and the twist map are such that

$$(X \otimes Y)(Z \otimes T) = (-1)^{(\deg Y)(\deg Z)} XZ \otimes YT, \quad \tau(X \otimes Y) = (-1)^{(\deg X)(\deg Y)} Y \otimes X, \quad (2.14)$$

for any homogeneous $X, Y, Z, T \in \mathcal{H}_q$.

Then, according to Definition 3.1 of II, a coloured Hopf superalgebra $(\mathcal{H}_q, +, \gamma_q, m_q, \iota_q, \Delta_{q,\nu}^{\lambda,\mu}, \epsilon_{q,\nu}, S_{q,\nu}^\mu; k, \mathcal{Q}, \mathcal{C}, \mathcal{G})$ is a coloured Hopf algebra, whose building blocks are Hopf superalgebras, and whose colour group elements σ^ν are superalgebra isomorphisms. In other words, the σ^ν 's satisfy Eqs. (I2.1)–(I2.4), and in addition are grade-preserving maps, i.e.,

$$\sigma^\nu \circ \gamma_q = \gamma_{q^\nu} \circ \sigma^\nu, \quad (2.15)$$

for any $q \in \mathcal{Q}$, and $\nu \in \mathcal{C}$. Such coloured Hopf superalgebras, which will still be denoted by \mathcal{H}^c , fulfil similar properties as ordinary coloured Hopf algebras, provided Eq. (2.14) is taken into account. In particular, if the \mathcal{H}_q 's are quasitriangular, then the coloured universal \mathcal{R} -matrix $\mathcal{R}^c = \{ \mathcal{R}_q^{\lambda,\mu} \mid q \in \mathcal{Q}, \lambda, \mu \in \mathcal{C} \}$ of $(\mathcal{H}^c, \mathcal{R}^c)$ satisfies the coloured graded YBE,

$$R_{q,12}^{\lambda,\mu} R_{q,13}^{\lambda,\nu} R_{q,23}^{\mu,\nu} = R_{q,23}^{\mu,\nu} R_{q,13}^{\lambda,\nu} R_{q,12}^{\lambda,\mu}, \quad (2.16)$$

which has the same form as the nongraded one, but wherein the first relation in Eq. (2.14) is used to evaluate the products.

Turning now to dual structures, let us assume that the Hopf superalgebras \mathcal{H}_q have Hopf duals $(\mathcal{H}_q^*, +, \tilde{\gamma}_q, \tilde{m}_q, \tilde{\iota}_q, \tilde{\Delta}_q, \tilde{\epsilon}_q, \tilde{S}_q; k)$ with corresponding grading maps denoted by $\tilde{\gamma}_q$, i.e.,

$$\tilde{\gamma}_q(x) = (-1)^{\deg x} x, \quad (2.17)$$

for any homogeneous $x \in \mathcal{H}_q^*$, and $\deg x = 0$ or 1 for x even or odd, respectively. The doubly nondegenerate bilinear forms $\langle \cdot, \cdot \rangle_q$ not only satisfy Eqs. (2.1)–(2.3), but are also consistent or, in other words, fulfil the relation

$$\langle \tilde{\gamma}_q(x), X \rangle_q = \langle x, \gamma_q(X) \rangle_q, \quad (2.18)$$

for any homogeneous $x \in \mathcal{H}_q^*$, and $X \in \mathcal{H}_q$.

It is then straightforward to show

Proposition II.8 *In the case of Hopf superalgebras, the maps $\rho^\nu : \mathcal{H}_q^* \rightarrow \mathcal{H}_{q^\nu}^*$, defined in Eq. (2.4), preserve the grading of the \mathcal{H}_q^* 's, i.e.,*

$$\rho^\nu \circ \tilde{\gamma}_q = \tilde{\gamma}_{q^\nu} \circ \rho^\nu, \quad (2.19)$$

for any $q \in \mathcal{Q}$, $\nu \in \mathcal{C}$, in addition to satisfying Proposition II.2. Moreover the coloured multiplication, unit, and antipode maps $\tilde{m}_{q,\lambda,\mu}^\nu$, \tilde{l}_q^ν , $\tilde{S}_{q,\mu}^\nu$, defined in Eq. (2.7), satisfy Propositions II.4 and II.7, provided the counterpart of Eq. (2.14) for the duals \mathcal{H}_q^* is taken into account.

Finally, Definition II.6 can be extended to coloured Hopf duals of coloured Hopf superalgebras with $\mathcal{H}^{c*} = (\mathcal{H}_q^*, +, \tilde{\gamma}_q, \tilde{m}_{q,\lambda,\mu}^\nu, \tilde{l}_q^\nu, \tilde{\Delta}_q, \tilde{\epsilon}_q, \tilde{S}_{q,\mu}^\nu; k, \mathcal{Q}, \mathcal{C}, \mathcal{G})$.

III DUALS OF COLOURED QUEA'S OF LIE (SUPER)ALGEBRAS

Whenever the Hopf (super)algebras \mathcal{H}_q are QUEA's of Lie (super)algebras $U_q(g)$, their Hopf duals \mathcal{H}_q^* (if they exist) are Hopf (super)algebras of quantized functions on the corresponding Lie (super)groups, $\text{Fun}_q(G) = G_q$.

Let $\{X_A\}$ and $\{x^A\}$ be dual bases of \mathcal{H}_q and \mathcal{H}_q^* , respectively. Hence

$$\langle x^A, X_B \rangle_q = \delta_B^A. \quad (3.1)$$

The universal \mathcal{T} -matrix of G_q [14, 16] is defined as the element of $G_q \otimes U_q(g)$ given by

$$\mathcal{T}_q = \sum_A x^A \otimes X_A. \quad (3.2)$$

In terms of the bases $\{X_A\}$ and $\{x^A\}$, the duality relations (2.2), and (2.8), between $\mathcal{H}^c = U^c(g)$ and $\mathcal{H}^{c*} = G^c$ can be written as

$$\begin{aligned} m_q(X_A \otimes X_B) &= \sum_C f_{AB}^C(q) X_C, & 1_q = \iota_q(1) &= \sum_A g^A(q) X_A, \\ \Delta_{q,\nu}^{\lambda,\mu}(X_A) &= \sum_{BC} h_A^{BC}(q^\lambda, q^\mu, q^\nu) X_B \otimes X_C, & \epsilon_{q,\nu}(X_A) &= \tilde{g}_A(q^\nu), \\ S_{q,\nu}^\mu(X_A) &= \sum_B s_A^B(q^\mu, q^\nu) X_B, \end{aligned} \quad (3.3)$$

and

$$\begin{aligned} \tilde{m}_{q,\lambda,\mu}^\nu(x^B \otimes x^C) &= \sum_A h_A^{BC}(q^\lambda, q^\mu, q^\nu) x^A, & \tilde{l}_q^\nu(1) &= \sum_A \tilde{g}_A(q^\nu) x^A, \\ \tilde{\Delta}_q(x^C) &= \sum_{AB} f_{AB}^C(q) x^A \otimes x^B, & \tilde{\epsilon}_q(x^A) &= g^A(q), \\ \tilde{S}_{q,\mu}^\nu(x^B) &= \sum_A s_A^B(q^\mu, q^\nu) x^A, \end{aligned} \quad (3.4)$$

where the structure constants for the multiplication (resp. coloured comultiplication) in $U^c(g)$ become the structure constants for the comultiplication (resp. coloured multiplication) in G^c , and similarly for the unit and counit, or the antipodes. Note that here we denote by the same symbol basis elements X_A (resp. x^A) belonging to different members $\mathcal{H}_{q^\nu} = U_{q^\nu}(g)$ (resp. $\mathcal{H}_{q^\nu}^* = G_{q^\nu}$) of the Hopf algebra set \mathcal{H} (resp. \mathcal{H}^*) in order not to overload notation by adding an extra index referring to the corresponding algebra. This should cause no confusion since from the context, it is always clear to which Hopf algebra X_A (resp. x^A) belongs.

Let us now introduce

Definition III.1 *The set $\mathcal{T}^c = \{ \mathcal{T}_q^\lambda \mid q \in \mathcal{Q}, \lambda \in \mathcal{C} \}$, where $\mathcal{T}_q^\lambda \equiv \mathcal{T}_{q^\lambda}$ is defined in Eq. (3.2), is called the universal \mathcal{T} -matrix of G^c .*

Definition III.2 *For any $q \in \mathcal{Q}$, and any $\lambda, \mu, \nu, \alpha, \beta \in \mathcal{C}$, let $\mathcal{U}_q^{\lambda, \mu} \in G_{q^\lambda} \otimes U_{q^\mu}(g)$, $\mathcal{A}_q^{\alpha, \mu} \dot{\otimes} \mathcal{B}_q^{\beta, \mu} \in G_{q^\alpha} \otimes G_{q^\beta} \otimes U_{q^\mu}(g)$, and $\mathcal{A}_q^{\lambda, \alpha} \overset{\nu}{\dot{\otimes}} \mathcal{B}_q^{\mu, \beta} \in G_{q^\nu} \otimes U_{q^\alpha}(g) \otimes U_{q^\beta}(g)$ be defined by*

$$\begin{aligned} \mathcal{U}_q^{\lambda, \mu} &\equiv (\text{id} \otimes S_{q, \lambda}^\mu) (\mathcal{T}_q^\lambda), \\ \mathcal{A}_q^{\alpha, \mu} \dot{\otimes} \mathcal{B}_q^{\beta, \mu} &\equiv \sum_{ABCD} a_A^C b_B^D x^A \otimes x^B \otimes m_{q^\mu}(X_C \otimes X_D), \\ \mathcal{A}_q^{\lambda, \alpha} \overset{\nu}{\dot{\otimes}} \mathcal{B}_q^{\mu, \beta} &\equiv \sum_{ABCD} a_A^C b_B^D \tilde{m}_{q, \lambda, \mu}^\nu(x^A \otimes x^B) \otimes X_C \otimes X_D, \end{aligned} \quad (3.5)$$

where

$$\mathcal{A}_q^{\lambda, \mu} \equiv \sum_{AC} a_A^C x^A \otimes X_C, \quad \mathcal{B}_q^{\lambda, \mu} \equiv \sum_{BD} b_B^D x^B \otimes X_D, \quad (3.6)$$

with $a_A^C, b_B^D \in k$, denote any two elements of $G_{q^\lambda} \otimes U_{q^\mu}(g)$.

From Definitions III.1, III.2, Eqs. (3.3), (3.4), and the generalized antipode axiom for $\tilde{S}_{q, \mu}^\nu$, given in Eq. (2.10), it is straightforward to show

Proposition III.3 *The elements of the coloured universal \mathcal{T} -matrix of G^c satisfy the relations*

$$\begin{aligned} \mathcal{T}_q^\lambda \dot{\otimes} \mathcal{T}_q^\lambda &= (\tilde{\Delta}_{q^\lambda} \otimes \text{id}) (\mathcal{T}_q^\lambda), \\ \mathcal{T}_q^\lambda \overset{\nu}{\dot{\otimes}} \mathcal{T}_q^\mu &= (\text{id} \otimes \Delta_{q, \nu}^{\lambda, \mu}) (\mathcal{T}_q^\nu), \\ \mathcal{U}_q^{\lambda, \mu} &= (\tilde{S}_{q, \mu}^\lambda \otimes \text{id}) (\mathcal{T}_q^\mu), \\ (\tilde{m}_{q, \lambda, \mu}^\nu \otimes \text{id}) (\mathcal{T}_q^\lambda \dot{\otimes} \mathcal{U}_q^{\mu, \lambda}) &= (\tilde{m}_{q, \mu', \lambda}^\nu \otimes \text{id}) (\mathcal{U}_q^{\mu', \lambda} \dot{\otimes} \mathcal{T}_q^\lambda) = \tilde{t}_q^\nu(1) \otimes 1_{q^\lambda}, \\ (\text{id} \otimes m_{q^\lambda}) (\mathcal{T}_q^\lambda \overset{\nu}{\dot{\otimes}} \mathcal{U}_q^{\mu, \lambda}) &= (\text{id} \otimes m_{q^\lambda}) (\mathcal{U}_q^{\mu', \lambda} \overset{\nu}{\dot{\otimes}} \mathcal{T}_q^\lambda) = \tilde{t}_q^\nu(1) \otimes 1_{q^\lambda}. \end{aligned} \quad (3.7)$$

Remark. The first relation in Eq. (3.7) is a well-known property of the universal \mathcal{T} -matrix of G_{q^λ} [16], but the remaining relations are new. Whenever the colour parameters reduce to that characterizing the unit element of \mathcal{G} , they go over into corresponding relations for the universal \mathcal{T} -matrix, with $\mathcal{U}_q^{\lambda,\mu}$ becoming the element of $G_q \otimes U_q(g)$ denoted as \mathcal{T}_q^{-1} in Ref. [16].

With the purpose of generalizing the algebraic formulation of the RTT -relations [16], let us introduce the coloured counterparts of $\mathcal{T}_{q1} \equiv \sum_A x^A \otimes (X_A \otimes 1_q)$, and $\mathcal{T}_{q2} \equiv \sum_A x^A \otimes (1_q \otimes X_A)$:

Definition III.4 Let $\mathcal{T}_{q,1\mu}^\lambda \in G_{q^\lambda} \otimes U_{q^\lambda}(g) \otimes U_{q^\mu}(g)$, and $\mathcal{T}_{q,2\lambda}^\mu \in G_{q^\mu} \otimes U_{q^\lambda}(g) \otimes U_{q^\mu}(g)$ be defined by

$$\mathcal{T}_{q,1\mu}^\lambda \equiv \sum_A x^A \otimes (X_A \otimes 1_{q^\mu}), \quad \mathcal{T}_{q,2\lambda}^\mu \equiv \sum_B x^B \otimes (1_{q^\lambda} \otimes X_B), \quad (3.8)$$

where $\mathcal{T}_q^\lambda = \sum_A x^A \otimes X_A$, and $\mathcal{T}_q^\mu = \sum_B x^B \otimes X_B$ are the universal \mathcal{T} -matrices of G_{q^λ} and G_{q^μ} , respectively.

The searched for result is expressed in

Proposition III.5 If the coloured Hopf (super)algebra $U^c(g)$ has a coloured universal \mathcal{R} -matrix, then the coloured universal \mathcal{T} -matrix of the coloured Hopf dual G^c satisfies the relation

$$\left(\tilde{1}_{q^\nu} \otimes \mathcal{R}_q^{\lambda,\mu} \right) \left(\mathcal{T}_{q,1\mu}^\lambda \overset{\nu}{\cdot} \mathcal{T}_{q,2\lambda}^\mu \right) = \left(\mathcal{T}_{q,2\lambda}^\mu \overset{\nu}{\cdot} \mathcal{T}_{q,1\mu}^\lambda \right) \left(\tilde{1}_{q^\nu} \otimes \mathcal{R}_q^{\lambda,\mu} \right), \quad (3.9)$$

where $\overset{\nu}{\cdot}$ denotes a generalized multiplication such that

$$\mathcal{T}_{q,1\mu}^\lambda \overset{\nu}{\cdot} \mathcal{T}_{q,2\lambda}^\mu \equiv \sum_{AB} \tilde{m}_{q,\lambda,\mu}^\nu(x^A \otimes x^B) \otimes (X_A \otimes X_B), \quad (3.10)$$

$$\mathcal{T}_{q,2\lambda}^\mu \overset{\nu}{\cdot} \mathcal{T}_{q,1\mu}^\lambda \equiv \sum_{AB} \tilde{m}_{q,\mu,\lambda}^\nu(x^B \otimes x^A) \otimes (X_A \otimes X_B). \quad (3.11)$$

Proof. Comparing Eq. (3.10) with the last relation in Eq. (3.5), and taking Proposition III.3 into account, we successively get

$$\mathcal{T}_{q,1\mu}^\lambda \overset{\nu}{\cdot} \mathcal{T}_{q,2\lambda}^\mu = \mathcal{T}_q^\lambda \overset{\nu}{\otimes} \mathcal{T}_q^\mu = (\text{id} \otimes \Delta_{q,\nu}^{\lambda,\mu}) (\mathcal{T}_q^\nu). \quad (3.12)$$

Moreover, from Eqs. (3.10)–(3.12), we also obtain

$$\mathcal{T}_{q,2\lambda}^\mu \overset{\nu}{\cdot} \mathcal{T}_{q,1\mu}^\lambda = (\text{id} \otimes \tau) \left(\mathcal{T}_{q,1\lambda}^\mu \overset{\nu}{\cdot} \mathcal{T}_{q,2\mu}^\lambda \right) = (\text{id} \otimes \tau \circ \Delta_{q,\nu}^{\mu,\lambda}) (\mathcal{T}_q^\nu). \quad (3.13)$$

Equation (3.9) then directly follows from Eq. (I2.15), expressing that $U^c(g)$ is almost co-commutative. \blacksquare

To explicitly show that Proposition III.5 provides an algebraic formulation of the coloured RTT -relations, introduced by Basu-Mallick [12], one has to consider matrix representations of Eq. (3.9). For such a purpose, let us denote by $D_q^{(i)}$ matrix representations of $U_q(g)$ in some n_i -dimensional k -modules $V_q^{(i)}$, where index i distinguishes between inequivalent representations. We may now represent the elements of the coloured universal \mathcal{R} - and \mathcal{T} -matrices by

$$R_q^{\lambda(i),\mu(j)} \equiv \left(D_{q^\lambda}^{(i)} \otimes D_{q^\mu}^{(j)} \right) \left(\mathcal{R}_q^{\lambda,\mu} \right), \quad T_q^{\lambda(i)} \equiv \sum_A x^A D_{q^\lambda}^{(i)}(X_A), \quad (3.14)$$

which are $n_i n_j \times n_i n_j$ and $n_i \times n_i$ matrices, valued in k and G_{q^λ} , respectively. Such definitions can be extended to provide representations of the operators in Eqs. (3.8), (3.10), and (3.11), as $n_i n_j \times n_i n_j$ matrices valued in G_{q^λ} , G_{q^μ} , G_{q^ν} , and G_{q^ν} , respectively:

$$\begin{aligned} T_{q,1\mu(j)}^{\lambda(i)} &\equiv \sum_A x^A \left(D_{q^\lambda}^{(i)}(X_A) \otimes I^{(j)} \right), \\ T_{q,2\lambda(i)}^{\mu(j)} &\equiv \sum_B x^B \left(I^{(i)} \otimes D_{q^\mu}^{(j)}(X_B) \right), \\ T_{q,1\mu(j)}^{\lambda(i)} \cdot^\nu T_{q,2\lambda(i)}^{\mu(j)} &\equiv \sum_{AB} \tilde{m}_{q,\lambda,\mu}^\nu (x^A \otimes x^B) \left(D_{q^\lambda}^{(i)}(X_A) \otimes D_{q^\mu}^{(j)}(X_B) \right), \\ T_{q,2\lambda(i)}^{\mu(j)} \cdot^\nu T_{q,1\mu(j)}^{\lambda(i)} &\equiv \sum_{AB} \tilde{m}_{q,\mu,\lambda}^\nu (x^B \otimes x^A) \left(D_{q^\lambda}^{(i)}(X_A) \otimes D_{q^\mu}^{(j)}(X_B) \right). \end{aligned} \quad (3.15)$$

Here $I^{(i)}$, and $I^{(j)}$ denote $n_i \times n_i$, and $n_j \times n_j$ unit matrices.

In the case of QUEA's of Lie superalgebras, since the $D_q^{(i)}$'s are graded matrices, their tensor product has to be evaluated in conformity with Eq. (2.14). According to the usual convention [7], matrix multiplication of graded matrices is assumed to be the same as that of nongraded ones, whereas the tensor product of two graded matrices is defined by

$$(A \otimes B)_{ij,kl} \equiv (-1)^{\pi_k(\pi_j + \pi_l)} A_{i,k} B_{j,l}. \quad (3.16)$$

Here π_k denotes the \mathbb{Z}_2 -grade of the k th row or column of matrix A or B . In particular, if A is homogeneous, its degree is given by $\deg A = \pi_k + \pi_l$ for any row index k , and any column index l . Special cases of Eq. (3.16) are

$$(A \otimes I)_{ij,kl} = A_{i,k} \delta_{j,l}, \quad (I \otimes A)_{ij,kl} = (-1)^{\pi_i(\pi_j + \pi_l)} \delta_{i,k} A_{j,l}. \quad (3.17)$$

From Proposition III.5, it is now straightforward to obtain

Corollary III.6 *In the representation $D_{q^\lambda}^{(i)} \otimes D_{q^\mu}^{(j)}$ of $U_{q^\lambda}(g) \otimes U_{q^\mu}(g)$, Eq. (3.9) becomes*

$$R_q^{\lambda(i), \mu(j)} \left(T_{q, 1\mu(j)}^{\lambda(i)} \cdot^\nu T_{q, 2\lambda(i)}^{\mu(j)} \right) = \left(T_{q, 2\lambda(i)}^{\mu(j)} \cdot^\nu T_{q, 1\mu(j)}^{\lambda(i)} \right) R_q^{\lambda(i), \mu(j)}, \quad (3.18)$$

where the various matrices are defined in Eqs. (3.14), and (3.15).

Remark. If we consider, in particular, the defining representations of $U_{q^\lambda}(g)$ and $U_{q^\mu}(g)$, Equation (3.18) can be rewritten in a simplified notation as

$$R_q^{\lambda, \mu} \left(T_{q1}^\lambda \cdot^\nu T_{q2}^\mu \right) = \left(T_{q2}^\mu \cdot^\nu T_{q1}^\lambda \right) R_q^{\lambda, \mu}. \quad (3.19)$$

This relation is formally identical with the coloured RTT -relations

$$R_q^{\lambda, \mu} T_{q1}^\lambda T_{q2}^\mu = T_{q2}^\mu T_{q1}^\lambda R_q^{\lambda, \mu}, \quad (3.20)$$

introduced by Basu-Mallick [12]. The dependence upon ν in Eq. (3.19), which is absent in Eq. (3.20), only means that we may evaluate the latter in various dual Hopf algebras G_{q^ν} , with ν running over \mathcal{C} .

In the next two Sections, we shall proceed to illustrate the new concepts and results presented here on two simple, but nevertheless significant examples.

IV THE COLOURED TWO-PARAMETER QUEA $U^c(gl(2))$ AND ITS DUAL $Gl^c(2)$

In I, we constructed a coloured QUEA by starting from the two-parameter deformation of $U(gl(2))$ [22, 23] in the Burdík and Hellinger formulation [24]. On the other hand, Frons-dal and Galindo [14] analyzed the duality relationship between the pair of Hopf algebras $U_{p,q}(gl(2))$ and $Gl_{p,q}(2)$, and constructed the universal \mathcal{T} -matrix of the latter. It is therefore interesting to study how this duality picture can be extended to the coloured context. For such a purpose, it will prove convenient to use the approach to $U_{p,q}(gl(2))$ considered in Refs. [14, 20], instead of that of Ref. [24]. The resulting coloured QUEA will therefore differ from that constructed in I. One should remember in this respect that there exist many ways of transforming a given Hopf algebra into a coloured one, depending upon the set of generators and the colour group that are used.

A The dual Hopf algebras $U_{p,q}(gl(2))$ and $Gl_{p,q}(2)$

In the standard formulation [20, 22], the quantum algebra $U_{p,q}(gl(2))$ is defined as the algebra generated by the elements $\{J_0, J_\pm, Z\}$, subject to the relations

$$[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = [2J_0]_Q, \quad [Z, J_0] = [Z, J_\pm] = 0, \quad (4.1)$$

where as usual

$$[X]_t \equiv \frac{t^X - t^{-X}}{t - t^{-1}}, \quad (4.2)$$

and the following combinations of p, q are considered:

$$Q \equiv \sqrt{pq}, \quad P \equiv \sqrt{p/q}. \quad (4.3)$$

Whereas the algebraic relations (4.1) only depend upon the first of these combined parameters, the coalgebraic structure depends upon both of them.

To describe the duality relationship between $U_{p,q}(gl(2))$ and $Gl_{p,q}(2)$, one has to consider another set of $U_{p,q}(gl(2))$ generators $\{\hat{J}_0, \hat{J}_\pm, \hat{Z}\}$, defined in terms of the first one by [20]

$$\hat{J}_0 = J_0, \quad \hat{J}_\pm = J_\pm Q^{\mp J_0 - \frac{1}{2}} P^{Z - \frac{1}{2}}, \quad \hat{Z} = Z, \quad (4.4)$$

and satisfying the commutation relations

$$[\hat{J}_0, \hat{J}_\pm] = \pm \hat{J}_\pm, \quad [\hat{J}_+, \hat{J}_-] = P^{2\hat{Z}-1} [2\hat{J}_0]_Q, \quad [\hat{Z}, \hat{J}_0] = [\hat{Z}, \hat{J}_\pm] = 0. \quad (4.5)$$

The remaining Hopf algebra maps are

$$\begin{aligned} \Delta_{p,q}(\hat{J}_0) &= \hat{J}_0 \otimes 1 + 1 \otimes \hat{J}_0, & \Delta_{p,q}(\hat{Z}) &= \hat{Z} \otimes 1 + 1 \otimes \hat{Z}, \\ \Delta_{p,q}(\hat{J}_+) &= \hat{J}_+ \otimes Q^{-2\hat{J}_0} P^{2\hat{Z}} + 1 \otimes \hat{J}_+, & \Delta_{p,q}(\hat{J}_-) &= \hat{J}_- \otimes 1 + Q^{2\hat{J}_0} P^{2\hat{Z}} \otimes \hat{J}_-, \\ \epsilon_{p,q}(X) &= 0, & X &\in \{\hat{J}_0, \hat{J}_\pm, \hat{Z}\}, \\ S_{p,q}(\hat{J}_0) &= -\hat{J}_0, & S_{p,q}(\hat{Z}) &= -\hat{Z}, \\ S_{p,q}(\hat{J}_+) &= -\hat{J}_+ Q^{2\hat{J}_0} P^{-2\hat{Z}}, & S_{p,q}(\hat{J}_-) &= -Q^{-2\hat{J}_0} P^{-2\hat{Z}} \hat{J}_-, \end{aligned} \quad (4.6)$$

and the universal \mathcal{R} -matrix can be written as [20, 25]

$$\mathcal{R}_{p,q} = Q^{-2\hat{J}_0 \otimes \hat{J}_0} P^{2(\hat{Z} \otimes \hat{J}_0 - \hat{J}_0 \otimes \hat{Z})} \sum_{n=0}^{\infty} \frac{(1 - Q^2)^n P^n}{[n]_Q!} Q^{-n(n+1)/2} \hat{J}_+^n \otimes \hat{J}_-^n, \quad (4.7)$$

where $[n]_Q! \equiv [n]_Q [n-1]_Q \dots [1]_Q$ for $n \in \mathbb{N}^+$, and $[0]_Q! \equiv 1$.

In the 2×2 defining representation $D_{p,q}$ of $U_{p,q}(gl(2))$, given by

$$\begin{aligned} D_{p,q}(\hat{J}_0) &= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & D_{p,q}(\hat{Z}) &= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ D_{p,q}(\hat{J}_+) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & D_{p,q}(\hat{J}_-) &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \end{aligned} \quad (4.8)$$

the universal \mathcal{R} -matrix (4.7) is represented by the 4×4 matrix

$$R_{p,q} \equiv (D_{p,q} \otimes D_{p,q})(\mathcal{R}_{p,q}) = Q^{1/2} \begin{pmatrix} Q^{-1} & 0 & 0 & 0 \\ 0 & P^{-1} & Q^{-1} - Q & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & Q^{-1} \end{pmatrix}. \quad (4.9)$$

For the corresponding quantum group $Gl_{p,q}(2)$, the defining T -matrix is specified by [14, 20]

$$T_{p,q} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (4.10)$$

with the commutation relations

$$\begin{aligned} ab &= qba, & ac &= pca, & bd &= pdb, & cd &= qdc, \\ bc &= (p/q)cb, & ad - da &= (q - p^{-1})bc, \end{aligned} \quad (4.11)$$

following from the RTT -relations corresponding to the R -matrix (4.9). The coalgebra maps are given by

$$\tilde{\Delta}_{p,q}(T_{p,q}) = T_{p,q} \otimes T_{p,q} = \begin{pmatrix} a \otimes a + b \otimes c & a \otimes b + b \otimes d \\ c \otimes a + d \otimes c & c \otimes b + d \otimes d \end{pmatrix}, \quad \tilde{\epsilon}_{p,q}(T_{p,q}) = I. \quad (4.12)$$

The quantum determinant, defined by

$$\mathcal{D} \equiv ad - qbc = ad - pcb = da - p^{-1}bc = da - q^{-1}cb, \quad (4.13)$$

is a group-like element, i.e.,

$$\tilde{\Delta}_{p,q}(\mathcal{D}) = \mathcal{D} \otimes \mathcal{D}, \quad \tilde{\epsilon}_{p,q}(\mathcal{D}) = 1, \quad (4.14)$$

but is not central (except for $p = q$), since

$$\mathcal{D}a = a\mathcal{D}, \quad \mathcal{D}b = P^{-2}b\mathcal{D}, \quad \mathcal{D}c = P^2c\mathcal{D}, \quad \mathcal{D}d = d\mathcal{D}. \quad (4.15)$$

With the assumption that \mathcal{D} is invertible, $Gl_{p,q}(2)$ is endowed with an antipode map

$$\tilde{S}_{p,q}(T_{p,q}) = (T_{p,q})^{-1} = \mathcal{D}^{-1} \begin{pmatrix} d & -p^{-1}b \\ -pc & a \end{pmatrix} = \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix} \mathcal{D}^{-1}, \quad (4.16)$$

such that $\tilde{S}_{p,q}(\mathcal{D}) = \mathcal{D}^{-1}$. This completes the Hopf algebraic structure of $Gl_{p,q}(2)$.

Following Fronsdal and Galindo [14], one may consider a Gauss decomposition of the T -matrix (4.10),

$$T_{p,q} = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & \hat{d} \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, \quad (4.17)$$

where a and \hat{d} are assumed invertible, thence $\beta = a^{-1}b$, $\gamma = ca^{-1}$, $\hat{d} = d - ca^{-1}b$. By assuming that the algebra can be augmented with the logarithms of a and \hat{d} , and by using the maps

$$a = e^\alpha, \quad \hat{d} = e^{-\delta}, \quad (4.18)$$

and the reparametrization

$$P = e^\theta, \quad Q = e^\varphi, \quad (4.19)$$

the quantum determinant (4.13) is transformed into $\mathcal{D} = \exp(\alpha - \delta)$, and the new variables $\{\alpha, \beta, \gamma, \delta\}$ are seen to satisfy the commutation relations of a solvable Lie algebra

$$\begin{aligned} [\alpha, \beta] &= (\varphi - \theta)\beta, & [\alpha, \gamma] &= (\varphi + \theta)\gamma, & [\alpha, \delta] &= 0, \\ [\delta, \beta] &= (\varphi + \theta)\beta, & [\delta, \gamma] &= (\varphi - \theta)\gamma, & [\beta, \gamma] &= 0, \end{aligned} \quad (4.20)$$

with a noncocommutative coproduct structure. As a consequence, $Gl_{p,q}(2)$ can be embedded into the enveloping algebra $U_{p,q}(\{\alpha, \beta, \gamma, \delta\})$ of such a Lie algebra. As shown by Fronsdal and Galindo [14], $U_{p,q}(\{\alpha, \beta, \gamma, \delta\})$ is dual to the QUEA $U_{p,q}(gl(2))$. Following common practice, we shall mean the larger enveloping algebra $U_{p,q}(\{\alpha, \beta, \gamma, \delta\})$ whenever speaking of the duality between $U_{p,q}(gl(2))$ and $Gl_{p,q}(2)$.

Dual bases of $U_{p,q}(gl(2))$ and $Gl_{p,q}(2)$ are given by [14]

$$X_A = \frac{Q^{a_1(a_1-1)/2} \hat{J}_-^{a_1}}{[a_1]_Q!} \frac{H^{a_2}}{a_2!} \frac{\tilde{H}^{a_3}}{a_3!} \frac{Q^{-a_4(a_4-1)/2} \hat{J}_+^{a_4}}{[a_4]_Q!}, \quad (4.21)$$

and

$$x^A = \gamma^{a_1} \alpha^{a_2} \delta^{a_3} \beta^{a_4}, \quad (4.22)$$

respectively, where

$$H \equiv \hat{J}_0 + \hat{Z}, \quad \tilde{H} \equiv \hat{J}_0 - \hat{Z}, \quad (4.23)$$

$A = (a_1, a_2, a_3, a_4)$, and $a_1, a_2, a_3, a_4 \in \mathbb{N}$. From Eqs. (3.2), (4.21), and (4.22), the universal \mathcal{T} -matrix of $Gl_{p,q}(2)$ can be written as

$$\mathcal{T}_{p,q} = \mathcal{E}xp_{Q^{-2}}(\gamma \hat{J}_-) \exp(\alpha H + \delta \tilde{H}) \mathcal{E}xp_{Q^2}(\beta \hat{J}_+), \quad (4.24)$$

in terms of the basic exponential function

$$\mathcal{E}xp_{t^2}(z) = \sum_{n=0}^{\infty} \frac{t^{-n(n-1)/2}}{[n]_t!} z^n. \quad (4.25)$$

In the next Subsection, it will prove convenient to consider another set of dual bases

$$\bar{X}_A = \sum_B c_A^B X_B = \frac{Q^{a_1(a_1-1)/2} \hat{J}_-^{a_1} \hat{J}_0^{a_2} \hat{Z}^{a_3} Q^{-a_4(a_4-1)/2} \hat{J}_+^{a_4}}{[a_1]_Q! a_2! a_3! [a_4]_Q!}, \quad (4.26)$$

$$\bar{x}^A = \sum_B (c^{-1})_B^A x^B = \gamma^{a_1} h^{a_2} \tilde{h}^{a_3} \beta^{a_4}. \quad (4.27)$$

Here, A has the same meaning as in Eqs. (4.21), (4.22),

$$h \equiv \alpha + \delta, \quad \tilde{h} \equiv \alpha - \delta, \quad (4.28)$$

and

$$c_A^B = 2^{-a_2-a_3} (c^{-1})_A^B = \delta_{a_1}^{b_1} \delta_{a_2+a_3}^{b_2+b_3} \delta_{a_4}^{b_4} 2^{-a_2-a_3} \sum_{t=\max(0, b_2-a_2)}^{\min(b_2, a_3)} (-1)^{a_3-t} \binom{b_2}{t} \binom{b_3}{a_3-t}, \quad (4.29)$$

where $\binom{s}{t}$ denotes a standard binomial coefficient. In terms of such dual bases, the universal \mathcal{T} -matrix (4.24) can be written as

$$\mathcal{T}_{p,q} = \mathcal{E}xp_{Q^{-2}}(\gamma \hat{J}_-) \exp(h \hat{J}_0 + \tilde{h} \hat{Z}) \mathcal{E}xp_{Q^2}(\beta \hat{J}_+). \quad (4.30)$$

B The coloured Hopf algebra $U^c(gl(2))$ and its dual $Gl^c(2)$

The defining relations (4.5) of the $U_{p,q}(gl(2))$ algebra are left invariant under the transformations

$$\sigma^\nu(\hat{J}_0) = \hat{J}_0, \quad \sigma^\nu(\hat{J}_\pm) = P^{(\nu-1)/2} \hat{J}_\pm, \quad \sigma^\nu(\hat{Z}) = \nu \hat{Z}, \quad (4.31)$$

where $\nu \in \mathcal{C} = \mathbb{C} \setminus \{0\}$, provided P is replaced by its ν th power, while Q is left unchanged, or equivalently [26]

$$(p, q) \rightarrow (p^{(\nu)}, q^{(\nu)}) = (p^{(1+\nu)/2} q^{(1-\nu)/2}, p^{(1-\nu)/2} q^{(1+\nu)/2}). \quad (4.32)$$

Hence, the σ^ν 's are isomorphic mappings between two $U_{p,q}(gl(2))$ algebras with different parameters, as given in Eq. (4.32), and they define a colour group $\mathcal{G} = Gl(1, \mathbb{C})$, since $\nu' \circ \nu = \nu' \nu$, $\nu^0 = 1$, and $\nu^i = \nu^{-1}$.

By choosing the parameter set $\mathcal{Q} = (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\})$, we therefore obtain a coloured quasitriangular Hopf algebra $U^c(gl(2))$, whose coloured comultiplication, counit, antipode, and \mathcal{R} -matrix are given by

$$\begin{aligned}
\Delta_{p,q,\nu}^{\lambda,\mu}(\hat{J}_0) &= \hat{J}_0 \otimes 1 + 1 \otimes \hat{J}_0, & \Delta_{p,q,\nu}^{\lambda,\mu}(\hat{Z}) &= \frac{\lambda}{\nu} \hat{Z} \otimes 1 + \frac{\mu}{\nu} 1 \otimes \hat{Z}, \\
\Delta_{p,q,\nu}^{\lambda,\mu}(\hat{J}_+) &= P^{(\lambda-\nu)/2} \hat{J}_+ \otimes Q^{-2\hat{J}_0} P^{2\mu\hat{Z}} + P^{(\mu-\nu)/2} 1 \otimes \hat{J}_+, \\
\Delta_{p,q,\nu}^{\lambda,\mu}(\hat{J}_-) &= P^{(\lambda-\nu)/2} \hat{J}_- \otimes 1 + P^{(\mu-\nu)/2} Q^{2\hat{J}_0} P^{2\lambda\hat{Z}} \otimes \hat{J}_-, \\
\epsilon_{p,q,\nu}(X) &= 0, & X &\in \{\hat{J}_0, \hat{J}_\pm, \hat{Z}\}, \\
S_{p,q,\nu}^\mu(\hat{J}_0) &= -\hat{J}_0, & S_{p,q,\nu}^\mu(\hat{J}_+) &= -P^{(\mu-\nu)/2} \hat{J}_+ Q^{2\hat{J}_0} P^{-2\mu\hat{Z}}, \\
S_{p,q,\nu}^\mu(\hat{J}_-) &= -P^{(\mu-\nu)/2} Q^{-2\hat{J}_0} P^{-2\mu\hat{Z}} \hat{J}_-, & S_{p,q,\nu}^\mu(\hat{Z}) &= -\frac{\mu}{\nu} \hat{Z}, \\
\mathcal{R}_{p,q}^{\lambda,\mu} &= Q^{-2\hat{J}_0 \otimes \hat{J}_0} P^{2(\lambda\hat{Z} \otimes \hat{J}_0 - \mu\hat{J}_0 \otimes \hat{Z})} \\
&\quad \times \sum_{n=0}^{\infty} \frac{(1-Q^2)^n}{[n]_Q!} Q^{-n(n+1)/2} P^{(\lambda+\mu)n/2} \hat{J}_+^n \otimes \hat{J}_-^n,
\end{aligned} \tag{4.33}$$

respectively.

In the defining representation (4.8), the coloured universal \mathcal{R} -matrix is represented by the 4×4 matrix

$$R_{p,q}^{\lambda,\mu} = Q^{1/2} \begin{pmatrix} Q^{-1} P^{(\lambda-\mu)/2} & 0 & 0 & 0 \\ 0 & P^{-(\lambda+\mu)/2} & Q^{-1} - Q & 0 \\ 0 & 0 & P^{(\lambda+\mu)/2} & 0 \\ 0 & 0 & 0 & Q^{-1} P^{-(\lambda-\mu)/2} \end{pmatrix}, \tag{4.34}$$

which is a matrix solution of the coloured YBE (I1.5), and gives back matrix (4.9) whenever $\lambda, \mu \rightarrow 1$.

Under transformation (4.31), the basis elements \overline{X}_A of $U_{p,q}(gl(2))$, defined in Eq. (4.26), become

$$\sigma^\nu(\overline{X}_A) = P^{(\nu-1)(a_1+a_4)/2} \nu^{a_3} \overline{X}_A. \tag{4.35}$$

Hence, from Eqs. (2.4) and (3.1), it follows that

$$\rho^\nu(\overline{x}^A) = P^{(1-\nu)(a_1+a_4)/2} \nu^{-a_3} \overline{x}^A \tag{4.36}$$

defines an isomorphic mapping between the $Gl_{p,q}(2)$ coalgebras whose parameters are given in Eq. (4.32). By taking Eqs. (4.27) and (4.28) into account, and by summing both sides of Eq. (4.36) over a_2 and a_3 , we also obtain the relation

$$\rho^\nu \left(\gamma^{a_1} e^{r(\alpha+\delta)} e^{s(\alpha-\delta)} \beta^{a_4} \right) = P^{(1-\nu)(a_1+a_4)/2} \gamma^{a_1} e^{r(\alpha+\delta)} e^{s(\alpha-\delta)/\nu} \beta^{a_4}, \quad (4.37)$$

valid for any $a_1, a_4 \in \mathbb{N}$, and any $r, s \in \mathbb{C}$.

The action of ρ^ν , and of its inverse ρ_ν , on $\alpha, \beta, \gamma, \delta$,

$$\begin{aligned} \rho^\nu(\alpha) &= \frac{\nu+1}{2\nu} \alpha + \frac{\nu-1}{2\nu} \delta, & \rho^\nu(\beta) &= P^{(1-\nu)/2} \beta, \\ \rho^\nu(\gamma) &= P^{(1-\nu)/2} \gamma, & \rho^\nu(\delta) &= \frac{\nu-1}{2\nu} \alpha + \frac{\nu+1}{2\nu} \delta, \end{aligned} \quad (4.38)$$

$$\begin{aligned} \rho_\nu(\alpha) &= \frac{1+\nu}{2} \alpha + \frac{1-\nu}{2} \delta, & \rho_\nu(\beta) &= P^{(\nu-1)/2} \beta, \\ \rho_\nu(\gamma) &= P^{(\nu-1)/2} \gamma, & \rho_\nu(\delta) &= \frac{1-\nu}{2} \alpha + \frac{1+\nu}{2} \delta, \end{aligned} \quad (4.39)$$

and on a, b, c, d, \mathcal{D} ,

$$\begin{aligned} \rho^\nu(a) &= \mathcal{D}^{(1-\nu)/(2\nu)} a, & \rho^\nu(b) &= (P^\nu \mathcal{D})^{(1-\nu)/(2\nu)} b, \\ \rho^\nu(c) &= (P^{-\nu} \mathcal{D})^{(1-\nu)/(2\nu)} c, & \rho^\nu(d) &= \mathcal{D}^{(1-\nu)/(2\nu)} d, & \rho^\nu(\mathcal{D}) &= \mathcal{D}^{1/\nu}, \end{aligned} \quad (4.40)$$

$$\begin{aligned} \rho_\nu(a) &= \mathcal{D}^{(\nu-1)/2} a, & \rho_\nu(b) &= (P \mathcal{D})^{(\nu-1)/2} b, \\ \rho_\nu(c) &= (P^{-1} \mathcal{D})^{(\nu-1)/2} c, & \rho_\nu(d) &= \mathcal{D}^{(\nu-1)/2} d, & \rho_\nu(\mathcal{D}) &= \mathcal{D}^\nu, \end{aligned} \quad (4.41)$$

can be obtained as special cases of Eqs. (4.36), and (4.37), respectively [27].

By using Eqs. (2.7), (4.11), (4.15), (4.16), (4.17), (4.18), (4.37), and (4.41), it is straightforward to determine the coloured maps of $Gl^c(2)$. The results are listed in Appendix A.

We shall now proceed to show that the relations obtained for $\tilde{m}_{p,q,\lambda,\mu}^\nu$ can be rewritten in an alternative way. For such a purpose, in analogy with Eq. (3.15), let us define

$$x(\lambda) \cdot^\nu y(\mu) \equiv \tilde{m}_{p,q,\lambda,\mu}^\nu(x \otimes y), \quad (4.42)$$

where to avoid ambiguity, we specify the algebra to which each element belongs by the corresponding colour parameter, e.g., $x(\lambda) \in Gl_{p(\lambda),q(\lambda)}(2)$.

The sixteen relations expressing $\tilde{m}_{p,q,\lambda,\mu}^\nu(x \otimes y)$, $x, y \in \{a, b, c, d\}$, as elements of $Gl_{p(\nu),q(\nu)}(2)$, and given in Eqs. (A1), (A5), can be combined into two different sets.

The first one contains the relations

$$\begin{aligned}
a(\lambda) \cdot^\nu b(\mu) &= q^{(\lambda)} b(\mu) \cdot^\nu a(\lambda), & a(\lambda) \cdot^\nu c(\mu) &= p^{(\lambda)} c(\mu) \cdot^\nu a(\lambda), \\
b(\lambda) \cdot^\nu d(\mu) &= p^{(\mu)} d(\mu) \cdot^\nu a(\lambda), & c(\lambda) \cdot^\nu d(\mu) &= q^{(\mu)} d(\mu) \cdot^\nu c(\lambda), \\
b(\lambda) \cdot^\nu c(\mu) &= \left(\frac{p^{(\lambda)} p^{(\mu)}}{q^{(\lambda)} q^{(\mu)}} \right)^{1/2} c(\mu) \cdot^\nu b(\lambda), \\
a(\lambda) \cdot^\nu d(\mu) - d(\mu) \cdot^\nu a(\lambda) &= \left\{ \left(q^{(\lambda)} q^{(\mu)} \right)^{1/2} - \left(p^{(\lambda)} p^{(\mu)} \right)^{-1/2} \right\} b(\lambda) \cdot^\nu c(\mu),
\end{aligned} \tag{4.43}$$

giving back the $Gl_{p,q}(2)$ defining relations (4.11) in the $\lambda, \mu, \nu \rightarrow 1$ limit, while the second set is given by the equations

$$\begin{aligned}
a(\lambda) \cdot^\nu a(\mu) &= a(\mu) \cdot^\nu a(\lambda), & a(\lambda) \cdot^\nu b(\mu) &= P^{(\mu-\lambda)/2} a(\mu) \cdot^\nu b(\lambda), \\
a(\lambda) \cdot^\nu c(\mu) &= P^{(\lambda-\mu)/2} a(\mu) \cdot^\nu c(\lambda), & a(\lambda) \cdot^\nu d(\mu) &= a(\mu) \cdot^\nu d(\lambda), \\
b(\lambda) \cdot^\nu b(\mu) &= P^{(\mu-\lambda)} b(\mu) \cdot^\nu b(\lambda), & b(\lambda) \cdot^\nu c(\mu) &= b(\mu) \cdot^\nu c(\lambda), \\
b(\lambda) \cdot^\nu d(\mu) &= P^{(\mu-\lambda)/2} b(\mu) \cdot^\nu d(\lambda), & c(\lambda) \cdot^\nu c(\mu) &= P^{\lambda-\mu} c(\mu) \cdot^\nu c(\lambda), \\
c(\lambda) \cdot^\nu d(\mu) &= P^{(\lambda-\mu)/2} c(\mu) \cdot^\nu d(\lambda), & d(\lambda) \cdot^\nu d(\mu) &= d(\mu) \cdot^\nu d(\lambda),
\end{aligned} \tag{4.44}$$

which have no standard counterpart.

Similarly, the relations for the coloured multiplication involving the quantum determinant, given in Eqs. (A2), (A3), (A4), (A6), and (A7), can be rewritten as

$$\begin{aligned}
\mathcal{D}(\lambda) \cdot^\nu a(\mu) &= a(\mu) \cdot^\nu \mathcal{D}(\lambda), & \mathcal{D}(\lambda) \cdot^\nu b(\mu) &= P^{-2\lambda} b(\mu) \cdot^\nu \mathcal{D}(\lambda), \\
\mathcal{D}(\lambda) \cdot^\nu c(\mu) &= P^{2\lambda} c(\mu) \cdot^\nu \mathcal{D}(\lambda), & \mathcal{D}(\lambda) \cdot^\nu d(\mu) &= d(\mu) \cdot^\nu \mathcal{D}(\lambda),
\end{aligned} \tag{4.45}$$

and

$$\begin{aligned}
\mathcal{D}(\lambda) \cdot^\nu a(\mu) &= (\mathcal{D}(\nu))^{(\lambda-\mu)/(2\nu)} \left(\mathcal{D}(\mu) \cdot^\nu a(\lambda) \right), \\
\mathcal{D}(\lambda) \cdot^\nu b(\mu) &= P^{(\mu-\lambda)/2} (\mathcal{D}(\nu))^{(\lambda-\mu)/(2\nu)} \left(\mathcal{D}(\mu) \cdot^\nu b(\lambda) \right), \\
\mathcal{D}(\lambda) \cdot^\nu c(\mu) &= P^{(\lambda-\mu)/2} (\mathcal{D}(\nu))^{(\lambda-\mu)/(2\nu)} \left(\mathcal{D}(\mu) \cdot^\nu c(\lambda) \right), \\
\mathcal{D}(\lambda) \cdot^\nu d(\mu) &= (\mathcal{D}(\nu))^{(\lambda-\mu)/(2\nu)} \left(\mathcal{D}(\mu) \cdot^\nu d(\lambda) \right), \\
\mathcal{D}(\lambda) \cdot^\nu \mathcal{D}(\mu) &= \mathcal{D}(\mu) \cdot^\nu \mathcal{D}(\lambda),
\end{aligned} \tag{4.46}$$

where the first set leads to Eq. (4.15) in the $\lambda, \mu, \nu \rightarrow 1$ limit, while the second has again no standard counterpart.

The simplicity of Eqs. (4.43), and (4.44) should be stressed. It is also remarkable that these equations are entirely independent of the final Hopf algebra colour parameter ν . Such a property reflects the fact that the two sets of equations (4.43), (4.44) are equivalent to the single matrix equation (3.19), where $R_{p,q}^{\lambda,\mu}$ is given by Eq. (4.34), and $T_{p,q}^\lambda = T_{p^{(\lambda)},q^{(\lambda)}} = \begin{pmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{pmatrix}$. It can indeed be checked by expressing Eq. (3.19) in elementwise form that the resulting independent relations can be grouped together in two sets, given by Eqs. (4.43), and (4.44). Since, moreover, the coalgebraic structure of $Gl^c(2)$ coincides with that of each individual standard quantum group $Gl_{p,q}(2)$, for (p, q) running over \mathcal{Q} , we did show in a very explicit way that our coloured $Gl^c(2)$ is equivalent to the coloured extension of $Gl_{p,q}(2)$, defined through coloured RTT -relations, according to Basu-Mallick's prescription [12].

By the way, it is worth noting that since parameter Q is left unchanged by the colour group transformations, the elements $\mathcal{T}_{p,q}^\lambda$ of the coloured universal \mathcal{T} -matrix, considered in Definition III.1, have no explicit λ -dependence, so that they are all given by Eq. (4.30) with $\hat{J}_0, \hat{J}_\pm, \hat{Z} \in U_{p^{(\lambda)},q^{(\lambda)}}(gl(2))$, and $h, \tilde{h}, \beta, \gamma \in Gl_{p^{(\lambda)},q^{(\lambda)}}(2)$.

As a final point of this Section, let us mention that the coloured maps $\tilde{m}_{p,q,\lambda,\mu}^\nu, \tilde{l}_{p,q}^\nu$, and $\tilde{S}_{p,q,\mu}^\nu$ may be extended from $Gl^c(2)$ to the whole dual of $U^c(gl(2))$ by considering the enveloping algebras $U_{p,q}(\{\alpha, \beta, \gamma, \delta\})$ for (p, q) running over \mathcal{Q} . For the coloured multiplication, for instance, one finds that the sixteen relations giving $\tilde{m}_{p,q,\lambda,\mu}^\nu(x \otimes y)$, $x, y \in \{\alpha, \beta, \gamma, \delta\}$, can be combined into six generalized commutation relations

$$\begin{aligned} [\alpha(\lambda), \beta(\mu)]^\nu &= P^{(\mu-\nu)/2}(\varphi - \lambda\theta)\beta(\nu), & [\alpha(\lambda), \gamma(\mu)]^\nu &= P^{(\mu-\nu)/2}(\varphi + \lambda\theta)\gamma(\nu), \\ [\delta(\lambda), \beta(\mu)]^\nu &= P^{(\mu-\nu)/2}(\varphi + \lambda\theta)\beta(\nu), & [\delta(\lambda), \gamma(\mu)]^\nu &= P^{(\mu-\nu)/2}(\varphi - \lambda\theta)\gamma(\nu), \\ [\alpha(\lambda), \delta(\mu)]^\nu &= 0, & [\beta(\lambda), \gamma(\mu)]^\nu &= 0, \end{aligned} \tag{4.47}$$

where

$$[x(\lambda), y(\mu)]^\nu \equiv x(\lambda) \stackrel{\nu}{\cdot} y(\mu) - y(\mu) \stackrel{\nu}{\cdot} x(\lambda), \tag{4.48}$$

and ten additional relations

$$\begin{aligned} [\alpha(\lambda), \alpha(\mu)]^\nu &= [\beta(\lambda), \beta(\mu)]^\nu = [\gamma(\lambda), \gamma(\mu)]^\nu = [\delta(\lambda), \delta(\mu)]^\nu = 0, \\ \alpha(\lambda) \stackrel{\nu}{\cdot} \beta(\mu) &= P^{(\mu-\lambda)/2}(2\mu)^{-1} \left((\lambda + \mu)\alpha(\mu) \stackrel{\nu}{\cdot} \beta(\lambda) - (\lambda - \mu)\delta(\mu) \stackrel{\nu}{\cdot} \beta(\lambda) \right), \\ \gamma(\lambda) \stackrel{\nu}{\cdot} \alpha(\mu) &= P^{(\lambda-\mu)/2}(2\lambda)^{-1} \left((\lambda + \mu)\gamma(\mu) \stackrel{\nu}{\cdot} \alpha(\lambda) + (\lambda - \mu)\gamma(\mu) \stackrel{\nu}{\cdot} \delta(\lambda) \right), \end{aligned}$$

$$\begin{aligned}
\alpha(\lambda) \cdot^\nu \delta(\mu) &= \alpha(\mu) \cdot^\nu \delta(\lambda) + (\lambda - \mu)(\lambda + \mu)^{-1} \left(\alpha(\mu) \cdot^\nu \alpha(\lambda) - \delta(\mu) \cdot^\nu \delta(\lambda) \right), \\
\gamma(\lambda) \cdot^\nu \beta(\mu) &= \gamma(\mu) \cdot^\nu \beta(\lambda), \\
\delta(\lambda) \cdot^\nu \beta(\mu) &= P^{(\mu-\lambda)/2} (2\mu)^{-1} \left(-(\lambda - \mu) \alpha(\mu) \cdot^\nu \beta(\lambda) + (\lambda + \mu) \delta(\mu) \cdot^\nu \beta(\lambda) \right), \\
\gamma(\lambda) \cdot^\nu \delta(\mu) &= P^{(\lambda-\mu)/2} (2\lambda)^{-1} \left((\lambda - \mu) \gamma(\mu) \cdot^\nu \alpha(\lambda) + (\lambda + \mu) \gamma(\mu) \cdot^\nu \delta(\lambda) \right). \quad (4.49)
\end{aligned}$$

Whenever $\lambda, \mu, \nu \rightarrow 1$, the former set reproduces the defining relations (4.20) of $U_{p,q}(\{\alpha, \beta, \gamma, \delta\})$, whereas the latter has no standard counterpart. Let us stress that contrary to what happens in Eqs. (4.43) and (4.44), there appears an explicit ν -dependence in Eqs. (4.47), and (4.49).

V THE COLOURED TWO-PARAMETER QUEA $U^c(gl(1/1))$ AND ITS DUAL $Gl^c(1/1)$

One of the simplest examples of dual pairs made of a QUEA of a Lie superalgebra and of a quantum supergroup consists in the two-parameter deformations of $U(gl(1/1))$ and $Gl(1/1)$ [28]–[32], which find some interesting applications to the multivariable Alexander-Conway polynomial and the free fermion model [33], and, in another context, to the XY quantum chain in a magnetic field [34].

In II, a coloured QUEA was constructed by starting from the two-parameter deformation of $U(gl(1/1))$ in the Bednar *et al* formulation [30, 31]. Moreover, Chakrabarti and Jagannathan [21] recently extended the Fronsdaal and Galindo universal T-matrix formalism [14] to the pair of Hopf superalgebras $U_{p,q}(gl(1/1))$ and $Gl_{p,q}(1/1)$. So all the ingredients needed to build an example of coloured Hopf dual in the graded case are available. To solve such a problem, we shall use the approach to $U_{p,q}(gl(1/1))$ employed in Ref. [21], instead of that of Refs. [30, 31]; hence, the results for $U^c(gl(1/1))$ derived here will slightly differ from those obtained in II.

A The dual Hopf superalgebras $U_{p,q}(gl(1/1))$ and $Gl_{p,q}(1/1)$

The quantum superalgebra $U_{p,q}(gl(1/1))$ is defined [21] as the algebra generated by two even elements $\{J_0, Z\}$, and two odd ones $\{J_+, J_-\}$, subject to the relations

$$[J_0, J_\pm] = \pm J_\pm, \quad \{J_+, J_-\} = [2Z]_Q, \quad J_\pm^2 = 0, \quad [Z, J_0] = [Z, J_\pm] = 0, \quad (5.1)$$

where $[X]_t$, and Q , P are still defined by Eqs. (4.2) and (4.3), respectively. As in the $U_{p,q}(gl(2))$ case, the coalgebraic structure depends upon both P and Q .

The duality relationship between $U_{p,q}(gl(1/1))$ and $Gl_{p,q}(1/1)$ is most easily described in terms of another set of $U_{p,q}(gl(1/1))$ generators $\{\hat{J}_0, \hat{J}_\pm, \hat{Z}\}$, with \hat{J}_0 , \hat{Z} even, and \hat{J}_\pm odd. The new generators are defined in terms of the old ones by

$$\hat{J}_0 = J_0, \quad \hat{J}_\pm = J_\pm Q^{\pm Z} P^{Z-\frac{1}{2}}, \quad \hat{Z} = Z, \quad (5.2)$$

and satisfy the superalgebra (anti)commutation relations

$$\begin{aligned} [\hat{J}_0, \hat{J}_\pm] &= \pm \hat{J}_\pm, & \{\hat{J}_+, \hat{J}_-\} &= \frac{p^{2\hat{Z}} - q^{-2\hat{Z}}}{p - q^{-1}}, & \hat{J}_\pm^2 &= 0, \\ [\hat{Z}, \hat{J}_0] &= [\hat{Z}, \hat{J}_\pm] = 0. \end{aligned} \quad (5.3)$$

The remaining Hopf superalgebra maps are given by

$$\begin{aligned} \Delta_{p,q}(\hat{J}_0) &= \hat{J}_0 \otimes 1 + 1 \otimes \hat{J}_0, & \Delta_{p,q}(\hat{Z}) &= \hat{Z} \otimes 1 + 1 \otimes \hat{Z}, \\ \Delta_{p,q}(\hat{J}_+) &= \hat{J}_+ \otimes p^{2\hat{Z}} + 1 \otimes \hat{J}_+, & \Delta_{p,q}(\hat{J}_-) &= \hat{J}_- \otimes 1 + q^{-2\hat{Z}} \otimes \hat{J}_-, \\ \epsilon_{p,q}(X) &= 0, & X &\in \{\hat{J}_0, \hat{J}_\pm, \hat{Z}\}, \\ S_{p,q}(\hat{J}_0) &= -\hat{J}_0, & S_{p,q}(\hat{Z}) &= -\hat{Z}, \\ S_{p,q}(\hat{J}_+) &= -p^{-2\hat{Z}} \hat{J}_+, & S_{p,q}(\hat{J}_-) &= -q^{2\hat{Z}} \hat{J}_-, \end{aligned} \quad (5.4)$$

and the universal \mathcal{R} -matrix can be written as [31, 32]

$$\mathcal{R}_{p,q} = p^{2\hat{Z} \otimes \hat{J}_0} q^{2\hat{J}_0 \otimes \hat{Z}} \left\{ 1 \otimes 1 - (p - q^{-1}) \hat{J}_+ \otimes \hat{J}_- \right\}. \quad (5.5)$$

In the 2×2 defining representation $D_{p,q}$ of $U_{p,q}(gl(1/1))$, given by Eq. (4.8), the universal \mathcal{R} -matrix (5.5) is represented by the 4×4 matrix

$$R_{p,q} \equiv (D_{p,q} \otimes D_{p,q})(\mathcal{R}_{p,q}) = \begin{pmatrix} Q & 0 & 0 & 0 \\ 0 & P^{-1} & Q - Q^{-1} & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & Q^{-1} \end{pmatrix}. \quad (5.6)$$

In deriving Eq. (5.6), use is made of Eq. (3.16) with the \mathbb{Z}_2 -grade of the first (resp. second) row or column of $D_{p,q}(X)$ defined as zero (resp. one).

The defining T -matrix of the corresponding quantum supergroup $Gl_{p,q}(1/1)$ is still given by Eq. (4.10), where a , d , and b , c are now even and odd, respectively, and satisfy the relations

$$\begin{aligned} ab &= p^{-1}ba, & ac &= q^{-1}ca, & bd &= pdb, & cd &= qdc, \\ bc &= -(p/q)cb, & ad - da &= (q - p^{-1})bc, & b^2 &= c^2 = 0. \end{aligned} \quad (5.7)$$

The latter follow from the RTT -relations corresponding to the R -matrix (5.6), with convention (3.17) taken into account. The coalgebra maps are given by Eq. (4.12), while the antipode one is

$$\tilde{S}_{p,q}(T_{p,q}) = (T_{p,q})^{-1} = \begin{pmatrix} a^{-1} + a^{-1}bd^{-1}ca^{-1} & -a^{-1}bd^{-1} \\ -d^{-1}ca^{-1} & d^{-1} + d^{-1}ca^{-1}bd^{-1} \end{pmatrix}, \quad (5.8)$$

where a and d are assumed invertible. The quantum superdeterminant, defined by

$$\mathcal{D} \equiv ad^{-1} - bd^{-1}cd^{-1}, \quad (5.9)$$

is both central and group-like, with $\tilde{S}_{p,q}(\mathcal{D}) = \mathcal{D}^{-1} = da^{-1} + ba^{-1}ca^{-1}$.

Gauss decomposition (4.17) still holds, but now a and \hat{d} are invertible even elements, while β and γ are odd. The antipode map and the quantum superdeterminant may be rewritten as

$$\tilde{S}_{p,q}(T_{p,q}) = \begin{pmatrix} a^{-1} - p^{-1}qca^{-2}\hat{d}^{-1}b & -p^{-1}a^{-1}\hat{d}^{-1}b \\ -qca^{-1}\hat{d}^{-1} & \hat{d}^{-1} \end{pmatrix}, \quad (5.10)$$

and

$$\mathcal{D} = a\hat{d}^{-1}, \quad (5.11)$$

respectively.

By assuming that the superalgebra can be augmented with the logarithms of a and \hat{d} , and by using the maps

$$a = e^\alpha, \quad \hat{d} = e^\delta, \quad (5.12)$$

and the reparametrization

$$p = e^{-\omega}, \quad q = e^{-\eta}, \quad (5.13)$$

the quantum superdeterminant is transformed into $\mathcal{D} = \exp(\alpha - \delta)$, and the new variables $\{\alpha, \beta, \gamma, \delta\}$ satisfy the (anti)commutation relations of a solvable Lie superalgebra

$$\begin{aligned} [\alpha, \beta] &= \omega\beta, & [\alpha, \gamma] &= \eta\gamma, & [\delta, \beta] &= \omega\beta, & [\delta, \gamma] &= \eta\gamma, \\ [\alpha, \delta] &= 0, & \{\beta, \gamma\} &= 0, & \beta^2 &= 0, & \gamma^2 &= 0, \end{aligned} \quad (5.14)$$

with a noncocommutative coproduct structure [21]. This shows that $Gl_{p,q}(1/1)$ can be embedded into the enveloping algebra $U_{p,q}(\{\alpha, \beta, \gamma, \delta\})$ of such a Lie superalgebra. According to Chakrabarti and Jagannathan [21], $U_{p,q}(\{\alpha, \beta, \gamma, \delta\})$ is dual to the QUEA $U_{p,q}(gl(1/1))$ [35].

Dual bases of $U_{p,q}(gl(1/1))$ and $Gl_{p,q}(1/1)$ are given by [21]

$$X_A = \hat{J}_-^{a_1} \frac{H^{a_2}}{a_2!} \frac{\tilde{H}^{a_3}}{a_3!} \hat{J}_+^{a_4}, \quad x^A = \gamma^{a_1} \alpha^{a_2} \delta^{a_3} \beta^{a_4}, \quad (5.15)$$

respectively, where $H \equiv \hat{Z} + \hat{J}_0$, $\tilde{H} \equiv \hat{Z} - \hat{J}_0$, $A = (a_1, a_2, a_3, a_4)$, $a_1, a_4 \in \{0, 1\}$, and $a_2, a_3 \in \mathbb{N}$. From Eqs. (3.2), and (5.15), the universal \mathcal{T} -matrix of $Gl_{p,q}(1/1)$ can be written as

$$\mathcal{T}_{p,q} = \exp(\gamma \hat{J}_-) \exp(\alpha H + \delta \tilde{H}) \exp(\beta \hat{J}_+), \quad (5.16)$$

in terms of standard exponentials.

In the next Subsection, it will prove convenient to consider another set of dual bases

$$\overline{X}_A = \sum_B c_A^B X_B = \hat{J}_-^{a_1} \frac{\hat{J}_0^{a_2}}{a_2!} \frac{\hat{Z}^{a_3}}{a_3!} \hat{J}_+^{a_4}, \quad (5.17)$$

$$\overline{x}^A = \sum_B (c^{-1})_B^A x^B = \gamma^{a_1} h^{a_2} \tilde{h}^{a_3} \beta^{a_4}, \quad (5.18)$$

where A has the same meaning as in Eq. (5.15),

$$h \equiv \alpha - \delta, \quad \tilde{h} \equiv \alpha + \delta, \quad (5.19)$$

and

$$\begin{aligned} c_A^B &= 2^{-a_2-a_3} (-1)^{a_2-b_2} (c^{-1})_A^B \\ &= \delta_{a_1}^{b_1} \delta_{a_2+a_3}^{b_2+b_3} \delta_{a_4}^{b_4} 2^{-a_2-a_3} \sum_{t=\max(0, b_2-a_2)}^{\min(b_2, a_3)} (-1)^{a_2-b_2+t} \binom{b_2}{t} \binom{b_3}{a_3-t}. \end{aligned} \quad (5.20)$$

In terms of them, the universal \mathcal{T} -matrix (5.16) can be rewritten as

$$\mathcal{T}_{p,q} = \exp(\gamma \hat{J}_-) \exp(h \hat{J}_0 + \tilde{h} \hat{Z}) \exp(\beta \hat{J}_+). \quad (5.21)$$

B The coloured Hopf superalgebra $U^c(gl(1/1))$ and its dual $Gl^c(1/1)$

The defining relations (5.3) of the $U_{p,q}(gl(1/1))$ superalgebra are left invariant under the grade-preserving transformations

$$\sigma^\nu(\hat{J}_0) = \hat{J}_0, \quad \sigma^\nu(\hat{J}_\pm) = \left(\frac{p^\nu - q^{-\nu}}{p - q^{-1}}\right)^{1/2} \hat{J}_\pm, \quad \sigma^\nu(\hat{Z}) = \nu \hat{Z}, \quad (5.22)$$

where $\nu \in \mathcal{C} = \mathbb{C} \setminus \{0\}$, provided p and q are replaced by their ν th powers, p^ν and q^ν (and the same for P and Q). The σ^ν 's are therefore isomorphic mappings between the $U_{p,q}(gl(1/1))$ and $U_{p^\nu, q^\nu}(gl(1/1))$ superalgebras, and define a colour group $\mathcal{G} = Gl(1, \mathbb{C})$.

By choosing the parameter set $\mathcal{Q} = (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\})$, we obtain a coloured quasitriangular Hopf superalgebra $U^c(gl(1/1))$, whose coloured comultiplication, counit, antipode, and \mathcal{R} -matrix are given by

$$\begin{aligned} \Delta_{p,q,\nu}^{\lambda,\mu}(\hat{J}_0) &= \hat{J}_0 \otimes 1 + 1 \otimes \hat{J}_0, & \Delta_{p,q,\nu}^{\lambda,\mu}(\hat{Z}) &= \frac{\lambda}{\nu} \hat{Z} \otimes 1 + \frac{\mu}{\nu} 1 \otimes \hat{Z}, \\ \Delta_{p,q,\nu}^{\lambda,\mu}(\hat{J}_+) &= A_\nu^\lambda \hat{J}_+ \otimes p^{2\mu\hat{Z}} + A_\nu^\mu 1 \otimes \hat{J}_+, & \Delta_{p,q,\nu}^{\lambda,\mu}(\hat{J}_-) &= A_\nu^\lambda \hat{J}_- \otimes 1 + A_\nu^\mu q^{-2\lambda\hat{Z}} \otimes \hat{J}_-, \\ \epsilon_{p,q,\nu}(X) &= 0, & X &\in \{\hat{J}_0, \hat{J}_\pm, \hat{Z}\}, \\ S_{p,q,\nu}^\mu(\hat{J}_0) &= -\hat{J}_0, & S_{p,q,\nu}^\mu(\hat{J}_+) &= -A_\nu^\mu p^{-2\mu\hat{Z}} \hat{J}_+, \\ S_{p,q,\nu}^\mu(\hat{J}_-) &= -A_\nu^\mu q^{2\mu\hat{Z}} \hat{J}_-, & S_{p,q,\nu}^\mu(\hat{Z}) &= -\frac{\mu}{\nu} \hat{Z}, \\ \mathcal{R}_{p,q}^{\lambda,\mu} &= p^{2\lambda\hat{Z} \otimes \hat{J}_0} q^{2\mu\hat{J}_0 \otimes \hat{Z}} \left\{ 1 \otimes 1 - \sqrt{(p^\lambda - q^{-\lambda})(p^\mu - q^{-\mu})} \hat{J}_+ \otimes \hat{J}_- \right\}, \end{aligned} \quad (5.23)$$

where $A_\nu^\lambda \equiv ((p^\lambda - q^{-\lambda})/(p^\nu - q^{-\nu}))^{1/2}$.

In the defining representation (4.8), the coloured universal \mathcal{R} -matrix is represented by the 4×4 matrix

$$R_{p,q}^{\lambda,\mu} = \begin{pmatrix} Q^{(\lambda+\mu)/2} P^{(\lambda-\mu)/2} & 0 & 0 & 0 \\ 0 & Q^{-(\lambda-\mu)/2} P^{-(\lambda+\mu)/2} & Q^{-(\lambda-\mu)/2} B^{\lambda,\mu} & 0 \\ 0 & 0 & Q^{(\lambda-\mu)/2} P^{(\lambda+\mu)/2} & 0 \\ 0 & 0 & 0 & Q^{-(\lambda+\mu)/2} P^{-(\lambda-\mu)/2} \end{pmatrix}, \quad (5.24)$$

with $B^{\lambda,\mu} \equiv \sqrt{(Q^\lambda - Q^{-\lambda})(Q^\mu - Q^{-\mu})}$. Such a matrix is a solution of the coloured graded YBE, and gives back matrix (5.6) whenever $\lambda, \mu \rightarrow 1$.

Under transformation (5.22), the basis elements \overline{X}_A of $U_{p,q}(gl(1/1))$, defined in Eq. (5.17), become

$$\sigma^\nu(\overline{X}_A) = (A_1^\nu)^{(a_1+a_4)/2} \nu^{a_3} \overline{X}_A. \quad (5.25)$$

Hence, from Eqs. (2.4) and (3.1), it follows that

$$\rho^\nu(\bar{x}^A) = (A_\nu^1)^{(a_1+a_4)/2} \nu^{-a_3} \bar{x}^A, \quad (5.26)$$

and

$$\rho^\nu(\gamma^{a_1} e^{r(\alpha-\delta)} e^{s(\alpha+\delta)} \beta^{a_4}) = (A_\nu^1)^{(a_1+a_4)/2} \gamma^{a_1} e^{r(\alpha-\delta)} e^{s(\alpha+\delta)/\nu} \beta^{a_4}, \quad (5.27)$$

where $r, s \in \mathbb{C}$.

As special cases of Eqs. (5.26) and (5.27), we obtain [27]

$$\begin{aligned} \rho^\nu(\alpha) &= \frac{1+\nu}{2\nu} \alpha + \frac{1-\nu}{2\nu} \delta, & \rho^\nu(\beta) &= A_\nu^1 \beta, \\ \rho^\nu(\gamma) &= A_\nu^1 \gamma, & \rho^\nu(\delta) &= \frac{1-\nu}{2\nu} \alpha + \frac{1+\nu}{2\nu} \delta, \end{aligned} \quad (5.28)$$

$$\begin{aligned} \rho_\nu(\alpha) &= \frac{\nu+1}{2} \alpha + \frac{\nu-1}{2} \delta, & \rho_\nu(\beta) &= A_1^\nu \beta, \\ \rho_\nu(\gamma) &= A_1^\nu \gamma, & \rho_\nu(\delta) &= \frac{\nu-1}{2} \alpha + \frac{\nu+1}{2} \delta, \end{aligned} \quad (5.29)$$

and

$$\begin{aligned} \rho^\nu(a) &= \mathcal{D}^{(\nu-1)/(2\nu)} a^{1/\nu}, & \rho^\nu(b) &= A_\nu^1 \mathcal{D}^{(\nu-1)/(2\nu)} a^{(1-\nu)/\nu} b, \\ \rho^\nu(c) &= q^{1-\nu} A_\nu^1 \mathcal{D}^{(\nu-1)/(2\nu)} a^{(1-\nu)/\nu} c, \\ \rho^\nu(d) &= \mathcal{D}^{(\nu-1)/(2\nu)} a^{(1-\nu)/\nu} \left\{ d - p \left(A_\nu^{\nu-1} \right)^2 c a^{-1} b \right\}, & \rho^\nu(\mathcal{D}) &= \mathcal{D}, \end{aligned} \quad (5.30)$$

$$\begin{aligned} \rho_\nu(a) &= \mathcal{D}^{(1-\nu)/2} a^\nu, & \rho_\nu(b) &= A_1^\nu \mathcal{D}^{(1-\nu)/2} a^{\nu-1} b, \\ \rho_\nu(c) &= q^{\nu-1} A_1^\nu \mathcal{D}^{(1-\nu)/2} a^{\nu-1} c, \\ \rho_\nu(d) &= \mathcal{D}^{(1-\nu)/2} a^{\nu-1} \left\{ d - p^\nu \left(A_1^{1-\nu} \right)^2 c a^{-1} b \right\}, & \rho_\nu(\mathcal{D}) &= \mathcal{D}. \end{aligned} \quad (5.31)$$

By proceeding as in Sec. IV, it is straightforward to determine the coloured maps of $Gl^c(1/1)$, listed in Appendix B, and to combine the results for the coloured multiplication $\tilde{m}_{p,q,\lambda,\mu}^\nu(x \otimes y)$, $x, y \in \{a, b, c, d\}$, given in Eqs. (B1)–(B4), and (B10), into two sets of eight relations each,

$$\begin{aligned} a(\lambda) \cdot^\nu b(\mu) &= p^{-\lambda} b(\mu) \cdot^\nu a(\lambda), & a(\lambda) \cdot^\nu c(\mu) &= q^{-\lambda} c(\mu) \cdot^\nu a(\lambda), \\ b(\lambda) \cdot^\nu d(\mu) &= p^\mu d(\mu) \cdot^\nu a(\lambda), & c(\lambda) \cdot^\nu d(\mu) &= q^\mu d(\mu) \cdot^\nu c(\lambda), \\ b(\lambda) \cdot^\nu c(\mu) &= -p^\mu q^{-\lambda} c(\mu) \cdot^\nu b(\lambda), & b(\lambda) \cdot^\nu b(\mu) &= c(\lambda) \cdot^\nu c(\mu) = 0, \\ a(\lambda) \cdot^\nu d(\mu) - d(\mu) \cdot^\nu a(\lambda) &= p^{-\mu} q^\lambda \sqrt{(p^\lambda - q^{-\lambda})(p^\mu - q^{-\mu})} b(\lambda) \cdot^\nu c(\mu), \end{aligned} \quad (5.32)$$

and

$$\begin{aligned}
a(\lambda) \cdot^\nu a(\mu) &= a(\mu) \cdot^\nu a(\lambda), & a(\lambda) \cdot^\nu b(\mu) &= A_\lambda^\mu a(\mu) \cdot^\nu b(\lambda), \\
a(\lambda) \cdot^\nu c(\mu) &= q^{\mu-\lambda} A_\lambda^\mu a(\mu) \cdot^\nu c(\lambda), \\
a(\lambda) \cdot^\nu d(\mu) - a(\mu) \cdot^\nu d(\lambda) &= q^{\lambda-\mu} A_\lambda^{\lambda-\mu} A_\mu^{\lambda-\mu} b(\lambda) \cdot^\nu c(\mu), \\
b(\lambda) \cdot^\nu c(\mu) &= (pq)^{\mu-\lambda} b(\mu) \cdot^\nu c(\lambda), & b(\lambda) \cdot^\nu d(\mu) &= p^{\mu-\lambda} A_\mu^\lambda b(\mu) \cdot^\nu d(\lambda), \\
c(\lambda) \cdot^\nu d(\mu) &= A_\mu^\lambda c(\mu) \cdot^\nu d(\lambda), & d(\lambda) \cdot^\nu d(\mu) &= d(\mu) \cdot^\nu d(\lambda).
\end{aligned} \tag{5.33}$$

Similarly, the relations for the coloured multiplication involving the quantum superdeterminant, given in Eqs. (B5)–(B9), and (B11), can be rewritten as

$$\mathcal{D}(\lambda) \cdot^\nu x(\mu) = x(\mu) \cdot^\nu \mathcal{D}(\lambda), \quad x \in \{a, b, c, d\}, \tag{5.34}$$

and

$$\begin{aligned}
\mathcal{D}(\lambda) \cdot^\nu a(\mu) &= (\mathcal{D}(\nu))^{(\lambda-\mu)/(2\nu)} (a(\nu))^{(\mu-\lambda)/\nu} \left(\mathcal{D}(\mu) \cdot^\nu a(\lambda) \right), \\
\mathcal{D}(\lambda) \cdot^\nu b(\mu) &= A_\lambda^\mu (\mathcal{D}(\nu))^{(\lambda-\mu)/(2\nu)} (a(\nu))^{(\mu-\lambda)/\nu} \left(\mathcal{D}(\mu) \cdot^\nu b(\lambda) \right), \\
\mathcal{D}(\lambda) \cdot^\nu c(\mu) &= q^{\mu-\lambda} A_\lambda^\mu (\mathcal{D}(\nu))^{(\lambda-\mu)/(2\nu)} (a(\nu))^{(\mu-\lambda)/\nu} \left(\mathcal{D}(\mu) \cdot^\nu c(\lambda) \right), \\
\mathcal{D}(\lambda) \cdot^\nu d(\mu) &= q^{\mu-\lambda} \left(A_{\lambda-\nu}^{\mu-\nu} \right)^2 (\mathcal{D}(\nu))^{(\lambda-\mu)/(2\nu)} (a(\nu))^{(\mu-\lambda)/\nu} \left(\mathcal{D}(\mu) \cdot^\nu d(\lambda) \right) \\
&\quad + p^{\mu-\nu} \left(A_{\lambda-\nu}^{\lambda-\mu} \right)^2 (\mathcal{D}(\nu))^{(3\nu-\mu)/(2\nu)} (a(\nu))^{(\mu-\lambda)/\nu} d(\nu), \\
\mathcal{D}(\lambda) \cdot^\nu \mathcal{D}(\mu) &= \mathcal{D}(\mu) \cdot^\nu \mathcal{D}(\lambda).
\end{aligned} \tag{5.35}$$

In the $\lambda, \mu, \nu \rightarrow 1$ limit, Eqs. (5.32), and (5.34) give back the $Gl_{p,q}(1/1)$ defining relations (5.7) and the central property of \mathcal{D} , while Eqs. (5.33), and (5.35) have no standard counterpart.

The two ν -independent sets of equations, contained in (5.32) and (5.33), can be explicitly shown to be equivalent to the single matrix equation (3.19), where $R_{p,q}^{\lambda,\mu}$ is given by Eq. (5.24). Moreover, as in Sec. IV, the elements $\mathcal{T}_{p,q}^\lambda$ of the coloured universal \mathcal{T} -matrix have no explicit λ -dependence.

The counterparts of Eqs. (4.47) and (4.49), for the coloured multiplication extended to the whole dual $U_{p,q}(\{\alpha, \beta, \gamma, \delta\})$ of $U^c(gl(1/1))$, are

$$\begin{aligned}
[\alpha(\lambda), \beta(\mu)]^\nu &= \lambda \omega A_\nu^\mu \beta(\nu), & [\alpha(\lambda), \gamma(\mu)]^\nu &= \lambda \eta A_\nu^\mu \gamma(\nu), \\
[\delta(\lambda), \beta(\mu)]^\nu &= \lambda \omega A_\nu^\mu \beta(\nu), & [\delta(\lambda), \gamma(\mu)]^\nu &= \lambda \eta A_\nu^\mu \gamma(\nu), \\
[\alpha(\lambda), \delta(\mu)]^\nu &= 0, & \{\beta(\lambda), \gamma(\mu)\}^\nu &= \beta(\lambda) \cdot^\nu \beta(\mu) = \gamma(\lambda) \cdot^\nu \gamma(\mu) = 0,
\end{aligned} \tag{5.36}$$

and

$$\begin{aligned}
[\alpha(\lambda), \alpha(\mu)]^\nu &= [\delta(\lambda), \delta(\mu)]^\nu = 0, \\
\alpha(\lambda) \cdot^\nu \beta(\mu) &= A_\lambda^\mu (2\mu)^{-1} \left((\lambda + \mu) \alpha(\mu) \cdot^\nu \beta(\lambda) + (\lambda - \mu) \delta(\mu) \cdot^\nu \beta(\lambda) \right), \\
\gamma(\lambda) \cdot^\nu \alpha(\mu) &= A_\mu^\lambda (2\lambda)^{-1} \left((\lambda + \mu) \gamma(\mu) \cdot^\nu \alpha(\lambda) - (\lambda - \mu) \gamma(\mu) \cdot^\nu \delta(\lambda) \right), \\
\alpha(\lambda) \cdot^\nu \delta(\mu) &= \alpha(\mu) \cdot^\nu \delta(\lambda) + (\lambda - \mu)(\lambda + \mu)^{-1} \left(-\alpha(\mu) \cdot^\nu \alpha(\lambda) + \delta(\mu) \cdot^\nu \delta(\lambda) \right), \\
\gamma(\lambda) \cdot^\nu \beta(\mu) &= \gamma(\mu) \cdot^\nu \beta(\lambda), \\
\delta(\lambda) \cdot^\nu \beta(\mu) &= A_\lambda^\mu (2\mu)^{-1} \left((\lambda - \mu) \alpha(\mu) \cdot^\nu \beta(\lambda) + (\lambda + \mu) \delta(\mu) \cdot^\nu \beta(\lambda) \right), \\
\gamma(\lambda) \cdot^\nu \delta(\mu) &= A_\mu^\lambda (2\lambda)^{-1} \left(-(\lambda - \mu) \gamma(\mu) \cdot^\nu \alpha(\lambda) + (\lambda + \mu) \gamma(\mu) \cdot^\nu \delta(\lambda) \right), \tag{5.37}
\end{aligned}$$

where

$$\{x(\lambda), y(\mu)\}^\nu \equiv x(\lambda) \cdot^\nu y(\mu) + y(\mu) \cdot^\nu x(\lambda). \tag{5.38}$$

Equation (5.36) gives back Eq. (5.14) whenever $\lambda, \mu, \nu \rightarrow 1$.

All the results obtained for the dual pair $(U^c(gl(1/1)), Gl^c(1/1))$ are therefore entirely similar to those derived for $(U^c(gl(2)), Gl^c(2))$ in Sec. IV.

VI CONCLUSION

In the present paper, we extended the notion of dually conjugate Hopf (super)algebras to the coloured Hopf (super)algebras \mathcal{H}^c , introduced in I and II. We showed that if the standard Hopf (super)algebras \mathcal{H}_q that are the building blocks of \mathcal{H}^c have Hopf duals \mathcal{H}_q^* , then the latter may be used to construct coloured Hopf duals \mathcal{H}^{c*} , endowed with coloured algebra and antipode maps, but with a standard coalgebraic structure.

Next, we reviewed the case where the \mathcal{H}_q 's are QUEA's of Lie (super)algebras $U_q(g)$, so that the \mathcal{H}_q^* 's are quantum (super)groups G_q . We extended the Fronsdaal and Galindo universal \mathcal{T} -matrix formalism [14] to the coloured pairs $(U^c(g), G^c)$ by introducing coloured universal \mathcal{T} -matrices. We then proved that the coloured RTT -relations, defining coloured $A(R)$ Hopf (super)algebras in Basu-Mallick's approach [12], may be considered as the realization in the $U_q(g)$ defining representation of a representation-free relation satisfied by the coloured universal \mathcal{R} - and \mathcal{T} -matrices.

Such results were finally illustrated by constructing two physically-relevant examples of coloured pairs, corresponding to the two-parameter deformations of $(U(gl(2)), Gl(2))$, and $(U(gl(1/1)), Gl(1/1))$, respectively.

In conclusion, we did prove that the formalism developed in I, II, and the present paper, provides an algebraic formulation of the coloured RTT -relations, and establishes a link between the coloured extensions of Drinfeld-Jimbo [5] and Faddeev-Reshetikhin-Takhtajan [8] pictures of quantum groups and quantum algebras. Since transfer matrices of quantum integrable models may be obtained from \mathcal{T} by specialization to given representations, we do think that the coloured extension of \mathcal{T} , and the related new algebraic structures introduced here, may have some interesting applications to such models.

There remain some open questions, which might be interesting topics for future study. We would like to mention here two of them. The first one consists in investigating the complementary approach to that considered in the present paper, namely trying to transform a set of quantum (super)groups G_q into a coloured Hopf (super)algebra \mathcal{H}^c by defining an appropriate colour group, then transferring the coloured structure to the duals $U_q(g)$ to build a coloured Hopf dual \mathcal{H}^{c*} . The second problem is to understand the relation, if any, between the latter and the coloured $U(R)$ Hopf (super)algebras introduced by Kundu and Basu-Mallick, and defined in terms of coloured RLL -relations [11].

APPENDIX A: COLOURED MAPS OF $Gl^c(2)$

In this Appendix, we list the results obtained for the coloured multiplication, unit, and antipode of $Gl^c(2)$. For their derivation, we used Eqs. (2.7), (4.11), (4.15), (4.16), (4.17), (4.18), (4.37), (4.41), and the fact that \bar{x}^{0000} is the unit of $Gl_{p,q}(2)$.

The coloured multiplication is given by

$$\tilde{m}_{p,q,\lambda,\mu}^\nu(x \otimes y) = P^{t_{\lambda,\mu}^\nu(x,y)/2} \mathcal{D}^{(\lambda+\mu-2\nu)/(2\nu)} xy, \quad (\text{A1})$$

$$\tilde{m}_{p,q,\lambda,\mu}^\nu(\mathcal{D} \otimes x) = P^{t_{\lambda,\mu}^\nu(\mathcal{D},x)/2} \mathcal{D}^{(2\lambda+\mu-\nu)/(2\nu)} x, \quad (\text{A2})$$

$$\tilde{m}_{p,q,\lambda,\mu}^\nu(x \otimes \mathcal{D}) = P^{t_{\lambda,\mu}^\nu(x,\mathcal{D})/2} \mathcal{D}^{(\lambda+2\mu-\nu)/(2\nu)} x, \quad (\text{A3})$$

$$\tilde{m}_{p,q,\lambda,\mu}^\nu(\mathcal{D} \otimes \mathcal{D}) = \mathcal{D}^{(\lambda+\mu)/\nu}, \quad (\text{A4})$$

where

$$\begin{aligned} t_{\lambda,\mu}^\nu(x, y) = & 0, \mu - \nu, -\mu + \nu, 0, \lambda + 2\mu - 3\nu, \lambda + 3\mu - 4\nu, \lambda + \mu - 2\nu, \\ & \lambda + 2\mu - 3\nu, -\lambda - 2\mu + 3\nu, -\lambda - \mu + 2\nu, -\lambda - 3\mu + 4\nu, \\ & -\lambda - 2\mu + 3\nu, 0, \mu - \nu, -\mu + \nu, 0, \end{aligned} \quad (\text{A5})$$

$$t_{\lambda,\mu}^\nu(\mathcal{D}, x) = 0, \mu - \nu, -\mu + \nu, 0, \quad (\text{A6})$$

$$t_{\lambda,\mu}^\nu(x, \mathcal{D}) = 0, \lambda + 4\mu - \nu, -\lambda - 4\mu + \nu, 0, \quad (\text{A7})$$

whenever x, y run over $\{a, b, c, d\}$, and are listed in lexicographical order.

For the coloured unit and antipode, the results read

$$\tilde{l}_{p,q}^\nu(1_k) = \tilde{l}_{p^{(\nu)},q^{(\nu)}}(1_k), \quad (\text{A8})$$

and

$$\tilde{S}_{p,q,\mu}^\nu(a) = \mathcal{D}^{-(\mu+\nu)/(2\nu)} d, \quad \tilde{S}_{p,q,\mu}^\nu(b) = -Q^{-1} (P^\nu \mathcal{D})^{-(\mu+\nu)/(2\nu)} b, \quad (\text{A9})$$

$$\tilde{S}_{p,q,\mu}^\nu(c) = -Q (P^{-\nu} \mathcal{D})^{-(\mu+\nu)/(2\nu)} c, \quad \tilde{S}_{p,q,\mu}^\nu(d) = \mathcal{D}^{-(\mu+\nu)/(2\nu)} a, \quad (\text{A10})$$

$$\tilde{S}_{p,q,\mu}^\nu(\mathcal{D}) = \mathcal{D}^{-\mu/\nu}, \quad (\text{A11})$$

respectively.

APPENDIX B: COLOURED MAPS OF $Gl^c(1/1)$

In this Appendix, we list the results obtained for the coloured multiplication, unit, and antipode of $Gl^c(1/1)$. For their derivation, we used Eqs. (2.7), (4.17), (5.7), (5.10), (5.12), (5.27), (5.31), and the fact that \bar{x}^{0000} is the unit of $Gl_{p,q}(1/1)$.

The coloured multiplication is given by

$$\tilde{m}_{p,q,\lambda,\mu}^\nu(x \otimes y) = C_{\lambda,\mu}^\nu(x, y) \mathcal{D}^{(2\nu-\lambda-\mu)/(2\nu)} a^{(\lambda+\mu-2\nu)/\nu} xy, \quad (B1)$$

$$\tilde{m}_{p,q,\lambda,\mu}^\nu(a \otimes d) = \mathcal{D}^{(2\nu-\lambda-\mu)/(2\nu)} a^{(\lambda+\mu-2\nu)/\nu} \left\{ ad - p^\mu q^{-\nu} \left(A_\nu^{\nu-\mu} \right)^2 cb \right\}, \quad (B2)$$

$$\begin{aligned} \tilde{m}_{p,q,\lambda,\mu}^\nu(d \otimes a) &= \mathcal{D}^{(2\nu-\lambda-\mu)/(2\nu)} a^{(\lambda+\mu-2\nu)/\nu} \left\{ da + p^\mu \left(q^{-\nu} \left(A_\nu^{\nu-\mu} \right)^2 \right. \right. \\ &\quad \left. \left. - p^\lambda \left(A_\nu^{2\nu-\lambda-\mu} \right)^2 \right) cb \right\}, \end{aligned} \quad (B3)$$

$$\tilde{m}_{p,q,\lambda,\mu}^\nu(d \otimes d) = \mathcal{D}^{(2\nu-\lambda-\mu)/(2\nu)} a^{(\lambda+\mu-2\nu)/\nu} \left\{ d^2 - p^{\lambda+\mu} \left(A_\nu^{2\nu-\lambda-\mu} \right)^2 ca^{-1} db \right\}, \quad (B4)$$

$$\tilde{m}_{p,q,\lambda,\mu}^\nu(\mathcal{D} \otimes x) = C_{\lambda,\mu}^\nu(\mathcal{D}, x) \mathcal{D}^{(3\nu-\mu)/(2\nu)} a^{(\mu-\nu)/\nu} x, \quad (B5)$$

$$\tilde{m}_{p,q,\lambda,\mu}^\nu(\mathcal{D} \otimes d) = \mathcal{D}^{(3\nu-\mu)/(2\nu)} a^{(\mu-\nu)/\nu} \left\{ d - p^\mu \left(A_\nu^{\nu-\mu} \right)^2 ca^{-1} b \right\}, \quad (B6)$$

$$\tilde{m}_{p,q,\lambda,\mu}^\nu(x \otimes \mathcal{D}) = C_{\lambda,\mu}^\nu(x, \mathcal{D}) \mathcal{D}^{(3\nu-\lambda)/(2\nu)} a^{(\lambda-\nu)/\nu} x, \quad (B7)$$

$$\tilde{m}_{p,q,\lambda,\mu}^\nu(d \otimes \mathcal{D}) = \mathcal{D}^{(3\nu-\lambda)/(2\nu)} a^{(\lambda-\nu)/\nu} \left\{ d - p^\lambda \left(A_\nu^{\nu-\lambda} \right)^2 ca^{-1} b \right\}, \quad (B8)$$

$$\tilde{m}_{p,q,\lambda,\mu}^\nu(\mathcal{D} \otimes \mathcal{D}) = \mathcal{D}^2. \quad (B9)$$

In Eq. (B1), x, y run over $\{a, b, c, d\}$ with the exceptions of $(x, y) = (a, d), (d, a), (d, d)$, and

$$\begin{aligned} C_{\lambda,\mu}^\nu(x, y) &= 1, A_\nu^\mu, q^{\mu-\nu} A_\nu^\mu, p^{\mu-\nu} A_\nu^\lambda, 1, (pq)^{\mu-\nu} A_\nu^\lambda A_\nu^\mu, p^{\mu-\nu} A_\nu^\lambda, q^{\lambda+\mu-2\nu} A_\nu^\lambda, \\ &\quad q^{\lambda+\mu-2\nu} A_\nu^\lambda A_\nu^\mu, 1, q^{\lambda+\mu-2\nu} A_\nu^\lambda, A_\nu^\mu, q^{\mu-\nu} A_\nu^\mu, \end{aligned} \quad (B10)$$

for (x, y) listed in lexicographical order. Similarly, in Eqs. (B5) and (B7), x runs over $\{a, b, c\}$, and

$$C_{\lambda,\mu}^\nu(\mathcal{D}, x) = 1, A_\nu^\mu, q^{\mu-\nu} A_\nu^\mu, \quad C_{\lambda,\mu}^\nu(x, \mathcal{D}) = 1, A_\nu^\lambda, q^{\lambda-\nu} A_\nu^\lambda. \quad (B11)$$

For the coloured unit and antipode, the results read

$$\tilde{l}_{p,q}^\nu(1_k) = \tilde{l}_{p^\nu, q^\nu}(1_k), \quad (B12)$$

and

$$\tilde{S}_{p,q,\mu}^\nu(a) = \mathcal{D}^{(\mu-\nu)/(2\nu)} \left\{ a^{-\mu/\nu} - p^{-\mu} q^\mu (A_\nu^\mu)^2 c a^{-(\mu+\nu)/\nu} \hat{d}^{-1} b \right\}, \quad (\text{B13})$$

$$\tilde{S}_{p,q,\mu}^\nu(b) = -p^{-\mu} A_\nu^\mu a^{-(\mu+\nu)/(2\nu)} \hat{d}^{-(\mu+\nu)/(2\nu)} b, \quad (\text{B14})$$

$$\tilde{S}_{p,q,\mu}^\nu(c) = -q^\mu A_\nu^\mu c a^{-(\mu+\nu)/(2\nu)} \hat{d}^{-(\mu+\nu)/(2\nu)}, \quad (\text{B15})$$

$$\tilde{S}_{p,q,\mu}^\nu(d) = \mathcal{D}^{-(\mu-\nu)/(2\nu)} \hat{d}^{-\mu/\nu}, \quad (\text{B16})$$

$$\tilde{S}_{p,q,\mu}^\nu(\mathcal{D}) = \mathcal{D}^{-1}, \quad (\text{B17})$$

respectively.

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